

# CARLEMAN ESTIMATES FOR GLOBAL UNIQUENESS, STABILITY AND NUMERICAL METHODS FOR COEFFICIENT INVERSE PROBLEMS

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**Abstract.** This is a review paper of the role of Carleman estimates in the theory of Multidimensional Coefficient Inverse Problems since the first inception of this idea in 1981.

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**1. Introduction.** This is a review paper of the Bukhgeim-Klibanov method (BK). The author considers this paper as an introductory material for BK. Because of many publications on BK, the author restricts himself to citations of those works which he is best familiar with. An interested reader can find further citations as well as analytical details in publications cited here. Three topics are discussed in this paper: (1) Global uniqueness for Multidimensional Coefficient Inverse Problems (MCIPs) with single measurement data via BK, (2) Global stability both for MCIPs and some ill-posed Cauchy problems via Carleman estimates, and (3) Related convergent numerical methods both for MCIPs and some ill-posed Cauchy problems.

BK was introduced in three originating papers of Bukhgeim and Klibanov in 1981 [35, 36, 74]. Until now BK remains the only technique enabling to prove global uniqueness and stability theorems for MCIPs with single measurement data. The state of the art in the field of Inverse Problems in 1981 is well reflected in the following citation from the paper [35]. “*Uniqueness theorems for multidimensional inverse problems have at present been obtained mainly in classes of piecewise analytic functions and similar classes or locally...Moreover, the technique of investigating these problems has, as a rule depended in an essential way on the type of the differential equation. In this note a new method of investigating inverse problems is proposed that is based on weighted a priori estimates. This method makes it possible to consider in a unified way a broad class of inverse problems for those equations  $Pu = f$  for which the solution of the Cauchy problem admits a Carleman estimate...The theorems of §1 were proved by M.V. Klibanov and those of §2 by A.L. Bukhgeim. They were obtained simultaneously and independently.*”

While the paper [35] contains announcement of results and an indication of the proof, papers [36, 74] contain first complete proofs. Also, see, e.g. [37, 48, 71, 75, 76, 77, 78, 82, 79, 84, 85, 86, 88, 94] and Sections 1.10 and 1.11 of the book [14] for some follow up publications of these authors on BK. Prior publications [35, 36, 74] only the so-called **local** uniqueness theorems were known for MCIPs with single measurement data. The term *local* means here that unknown spatially dependent coefficients were assumed to be either piecewise analytic functions, or functions represented via truncated Fourier-like series, or sufficiently small perturbations of constants. At that time the common desire of many mathematicians working on MCIPs was to prove **global** uniqueness theorems. That is, to prove uniqueness for the case when the unknown coefficient  $a(x)$ ,  $x \in \mathbb{R}^n$ ,  $n \geq 2$  satisfies only some natural conditions, such as, e.g.  $a \in C^k$ . However, it was unclear at that time how to do this. Indeed, standard methods were basically based on integral equations and did not work for this goal. The absence of global uniqueness theorems was the main stumbling block in the field of Inverse Problems in 1970s. This is what has originally motivated the author in 1979 to think about moving away from traditional techniques. The author has spent two years to figure out the solution.

This paper is focused only on MCIPs with single measurement data. BK is based on a special use of Carleman estimates for MCIPs. Roughly speaking, as soon as Carleman estimate is valid for a PDE operator, BK can be applied. Therefore, the generality of BK is due to the fact that Carleman estimates are valid for hyperbolic, parabolic, elliptic, the non-stationary Schrödinger and some other operators. Schematically, BK consists of the following two steps:

**Step 1.** Given an MCIP, figure out whether a proper Carleman estimate is valid for the PDE operator of this problem. If not, derive a proper Carleman estimate (if possible).

**Step 2.** Given a proper Carleman estimate, apply BK.

Carleman estimates were originated in 1939 in a remarkable work of a distinguished Swedish mathematician Torsten Carleman [42]. Since then and up to now they have been traditionally applied by many authors to proofs of uniqueness theorems for various ill-posed problems for PDEs with the Cauchy data on non-characteristic hypersurfaces, see, e.g. the paper of Calderon [40] and the book of Hörmander [57]. While in these references Carleman estimates were used for functions with compact support, the book of Lavrent’ev, Romanov and Shishatskii [112] (Chapter 4) uses functions with non-compact support. As a result, the technique of this book allowed to prove not only uniqueness but Hölder stability results as well for some ill-posed Cauchy problems. Thus, it was briefly noticed in earlier publications [76, 82] with the reference to

[112] that the applicability of BK to an MCIP usually automatically implies the Hölder stability estimate for this MCIP. Since the author is concerned with MCIPs, multiple quite interesting works on applications of Carleman estimates to various ill-posed Cauchy problems are not cited here.

BK applies Carleman estimates, in a specially designed way, to proofs of global uniqueness and stability results for MCIPs. MCIPs are substantially different from those Cauchy problems. Indeed, in such a Cauchy problem all coefficients of the corresponding PDE operator are known, the initial condition is unknown, and the Cauchy data at a non-characteristic hypersurface are known. It is required then to reconstruct the solution  $u$  of the corresponding PDE. On the other hand, in an MCIP at least one of coefficients,  $a(x)$ , of the corresponding PDE operator is unknown, the initial condition is known, and the lateral Cauchy data are known as well. It is required then to reconstruct the pair of functions  $(a, u)$ . Those Cauchy problem are linear and any MCIP is nonlinear.

The term “MCIP with single measurement data” means the problem of the recovery of one of coefficients of a PDE from a boundary measurement generated by a single set of initial conditions. In the case of either the point source or a plane wave this means either a single position of that source or a single direction of that incident plane wave. More generally, this is a single pair of initial conditions for a hyperbolic PDE and a single initial condition for a parabolic PDE. Sometimes a few initial sets of initial conditions are allowed. In the case when  $k$  coefficients are unknown,  $k$  sets of initial conditions are allowed, which means  $k$  measurements. MCIPs with single measurement are non-overdetermined ones. The non-overdetermination means that the number of free variables in the data equals the number of free variables in the unknown coefficient. The single measurement case is the one with the minimal amount of the available information. Hence, this is the most economical way of data collection. In particular, in military applications the single measurement case is far preferable to the case of many measurements. This is because an installation of each source carries a serious risk for life of soldiers on a battlefield.

There is a single condition of BK, which has been viewed as a drawback from the applied standpoint for a long time since 1981, This condition is still not lifted. Specifically, BK requires that at least one initial condition to be non-zero in the entire domain of interest. However, after getting an extensive recent numerical experience with the approximately globally convergent numerical method for MCIPs (see the book of Beilina and Klivanov [14] and Section 6 below), the author believes now that this drawback is an absolutely insignificant one precisely from the applied standpoint (although the mathematical question remains open). Indeed, the most interesting case in applications is the case when the initial condition is the function  $\delta(x - x_0)$  with a fixed position of the source  $x_0$ . However, replacement of this function by its approximation via a narrow Gaussian  $\delta_\varepsilon(x - x_0)$ ,

$$\delta_\varepsilon(x - x_0) = C_\varepsilon \exp\left(-\frac{|x - x_0|^2}{\varepsilon^2}\right), \int_{\mathbb{R}^n} \delta_\varepsilon(x - x_0) dx = 1$$

immediately lets BK working ( $\varepsilon > 0$  is sufficiently small here). The corresponding boundary data, which model the data resulting from a measurement, have only an insignificant change. Therefore, if a numerical method for a corresponding MCIP is stable, as it must be, then this change should affect the solution only insignificantly. Furthermore, physicists and engineers are indifferent to such a replacement because of the above reasons. This is why functions  $\delta(x - x_0)$  and  $\delta_\varepsilon(x - x_0)$  are equivalent precisely from the applied standpoint.

Still, the author has proved [94] uniqueness theorem for an MCIP for the equation  $u_{tt} = \Delta u + a(x, y, z)u$ ,  $(x, y, z) \in \mathbb{R}^3$  for the case of an incident plane wave with  $u(x, y, z, 0) = 0$ ,  $u_t(x, y, z, 0) = \delta(z)$  and under the assumption that this equation for the function  $u$  is written in finite differences with respect to  $(x, y)$ .

The idea of BK is described in Section 3. Five examples of this section show how BK works. In a less general form examples of Sections 3.2, 3.3.2 and 3.4 were first published in originating works [35, 74], also see [14, 82, 86] for more general forms of these three examples as well as for two other examples of Section 3. In principle, it is possible to formulate BK in a general abstract form, see, e.g. the earlier paper of the author [75] for this form. However, it is not necessary to do so for the understanding of BK. Previously

published relevant results are discussed in Section 4 as well as in the end of each of Sections 2,3,5,6.

## 2. Carleman Estimates, Hölder Stability and the Quasi-Reversibility Method.

**2.1. Definition of the Carleman estimate.** We now introduce the notion of the pointwise Carleman estimate for a general Partial Differential Operator of the second order. Let  $G \subset \mathbb{R}^n$  be a bounded domain with a piecewise smooth boundary  $\partial G$ . Let the function  $\xi \in C^2(\overline{G})$  and  $|\nabla \xi| \neq 0$  in  $\overline{G}$ . For a number  $c \geq 0$  denote

$$\xi_c = \{x \in \overline{G} : \xi(x) = c\}, G_c = \{x \in G : \xi(x) > c\}.$$

Assume that  $G_c \neq \emptyset$ . Let  $\Gamma_c \subseteq \partial G$  a part of the boundary  $\partial G$  defined as

$$\Gamma_c = \{x \in \partial G : \xi(x) \geq c\}.$$

Let  $G_c \neq \emptyset$ . Then the boundary of the domain  $G_c$  consists of two parts,

$$\partial G_c = \partial_1 G_c \cup \partial_2 G_c, \partial_1 G_c = \xi_c, \partial_2 G_c = \Gamma_c. \quad (2.1)$$

Let  $\lambda > 1$  be a parameter, which we will consider to be large. Consider the function  $\varphi_\lambda(x)$ ,

$$\varphi_\lambda(x) = \exp(\lambda \xi(x)). \quad (2.2)$$

It follows from (2.1), (2.2) that

$$\min_{\overline{G}_c} \varphi_\lambda(x) = \varphi_\lambda(x)|_{\xi_c} \equiv \exp(\lambda c). \quad (2.3)$$

Let  $A(x, D)$  be a linear Partial Differential Operator of the second order with real valued coefficients in  $G$  and with its principal part  $A_0(x, D)$ ,

$$A(x, D)u = \sum_{|\alpha| \leq 2} a_\alpha(x) D^\alpha u, \quad A_0(x, D)u = \sum_{|\alpha|=2} a_\alpha(x) D^\alpha u. \quad (2.4)$$

$$a_\alpha \in C^1(\overline{G}) \text{ for } |\alpha| = 2, K := \max_{|\alpha|=2} (\|a_\alpha\|_{C^1(\overline{G}_c)}); \quad a_\alpha \in C(\overline{G}) \text{ for } |\alpha| = 0, 1. \quad (2.5)$$

**Definition 2.1.** Let  $G_c \neq \emptyset$ . We say that the operator  $A_0(x, D)$  admits pointwise Carleman estimate in the domain  $G_c$  with the Carleman Weight Function (CWF)  $\varphi_\lambda(x)$  if there exist constants  $\lambda_0(\Omega, K) > 1, C_0 = C_0(\Omega, K) \geq 0, C(\Omega, K) > 0$  depending only on the domain  $G_c$  and the number  $K$ , such that the following a priori estimate holds

$$(A_0 u)^2 \varphi_\lambda^2(x) \geq \frac{C_0}{\lambda} \sum_{|\alpha|=2} (D^\alpha u)^2 \varphi_\lambda^2(x) + C\lambda (\nabla u)^2 \varphi_\lambda^2(x) + C\lambda^3 u^2 \varphi_\lambda^2(x) + \operatorname{div} U, \quad (2.6)$$

$$\forall \lambda \geq \lambda_0, \forall u \in C^2(\overline{G}), \forall x \in G_c. \quad (2.7)$$

In (2.6) the term under the divergence sign satisfies the following estimate

$$|U| \leq C\lambda^3 \left[ (\nabla u)^2 + u^2 \right] \varphi_\lambda^2(x) + \frac{C_0}{\lambda} \sum_{|\alpha|=2} (D^\alpha u)^2 \varphi_\lambda^2(x). \quad (2.8)$$

In the case of parabolic and elliptic operators  $C_0 > 0$  and  $C_0 = 0$  in the case of a hyperbolic operator. Lemma 2.1 is elementary.

**Lemma 2.1.** Let conditions (2.5) imposed on coefficients of the operator  $A$  be valid. Suppose that the Carleman estimate (2.6)-(2.8) is valid for the principal part  $A_0(x, D)$  of the operator  $A(x, D)$ . Then it is

also valid for the operator  $A(x, D)$ , although with a different constant  $\lambda_0$ . In other words, the Carleman estimate depends only on the principal part of the operator.

**Proof.** We have

$$(Au)^2 \varphi_\lambda^2(x) \geq (A_0 u)^2 \varphi_\lambda^2(x) - M \left[ (\nabla u)^2 + u^2 \right] \varphi_\lambda^2(x), \quad (2.9)$$

where  $M > 0$  is a constant depending only on the maximum of norms  $\|a_\alpha\|_{C(\overline{G})}$ ,  $|\alpha| = 0, 1$ . Substituting (2.9) in (2.6) and taking  $\lambda$  sufficiently large, we again obtain (2.6).  $\square$

**2.2. Hölder stability.** Consider the following Cauchy problem for the differential inequality

$$|A_0 u| \leq B(|\nabla u| + |u| + |f|), \quad \forall x \in G_c, \quad (2.10)$$

$$u|_{\Gamma_c} = g_0(x), \quad \partial_n u|_{\Gamma_c} = g_1(x), \quad (2.11)$$

where  $B = \text{const.} > 0$  and  $f \in L_2(G_c)$  is a function. Clearly, functions  $g_0, g_1$  in (2.11) are the Cauchy data for the function  $u$ . In particular, equation  $Au = f$  with the boundary data (2.11) can be reduced to the problem (2.10), (2.11). We want to estimate the function  $u$  via functions  $f, g_0, g_1$ . Such estimates were derived in Chapter 4 of the book of Lavrent'ev, Romanov and Shishatskii [112] for parabolic, elliptic and hyperbolic operators.

**Theorem 2.1** (Hölder stability estimate). *Assume that conditions (2.5) hold and that the Carleman estimate of Definition 2.1 is valid. Suppose that there exists a sufficiently small number  $\varepsilon > 0$  such that the domain  $G_{c+3\varepsilon} \neq \emptyset$ . Denote  $m = \max_{\overline{G_c}} \xi(x)$ . Define the number  $\beta = 2\varepsilon / (3m + 2\varepsilon) \in (0, 1)$ . Let functions  $g_0, g_1, f$  be such that  $g_0 \in H^1(\Gamma_c)$ ,  $g_1 \in L_2(\Gamma_c)$ ,  $f \in L_2(G_c)$ . Let the function  $u \in C^2(\overline{G_c})$  satisfies conditions (2.10), (2.11). Then there exists a sufficiently small number  $\delta_0 = \delta_0(\varepsilon, m, B, K, G_c) \in (0, 1)$  and a constant  $C_1 = C_1(\varepsilon, m, B, K, G_c) > 0$  such that if  $\delta \in (0, \delta_0)$ ,*

$$\|f\|_{L_2(G_c)} + \|g_0\|_{H^1(\Gamma_c)} + \|g_1\|_{L_2(\Gamma_c)} \leq \delta \quad (2.12)$$

then the following Hölder stability estimate holds

$$\|u\|_{H^1(G_{c+3\varepsilon})} \leq C_1 \left( 1 + \|u\|_{H^1(G_c)} \right) \delta^\beta, \quad \forall \delta \in (0, \delta_0), \quad (2.13)$$

**Remark 2.1.** Estimate (2.13) is Hölder stability estimate because  $\beta \in (0, 1)$ . If we would have  $\beta = 1$ , then (2.13) would become the Lipschitz stability estimate. Because of the presence of the term  $\|u\|_{H^1(G_c)}$ , this is the so-called “conditional stability estimate”, which are common in the theory of ill-posed problems [5, 14, 49, 70, 145]. Indeed, the presence of this term actually means that we assume an *a priori* given upper bound for the norm  $\|u\|_{H^1(G_c)}$ .

**Proof of Theorem 2.1.** In this proof  $C = C(\varepsilon, K, G_c)$  and  $C_1 = C_1(\varepsilon, m, B, K, G_c)$  denote different positive constants depending on listed parameters but independent on the function  $u$  and the parameter  $\lambda$ . Obviously  $G_{c+3\varepsilon} \subset G_{c+2\varepsilon} \subset G_{c+\varepsilon} \subset G_c$ . Since  $G_{c+3\varepsilon} \neq \emptyset$ , then  $G_{c+2\varepsilon}, G_{c+\varepsilon}, G_c \neq \emptyset$ . Let  $\chi(x)$  be a function such that

$$\chi \in C^2(\overline{G_c}), \quad \chi(x) = \begin{cases} 1, & x \in G_{c+2\varepsilon}, \\ 0, & x \in G_c \setminus G_{c+\varepsilon}, \\ \in [0, 1], & x \in G_{c+\varepsilon} \setminus G_{c+2\varepsilon}. \end{cases} \quad (2.14)$$

Consider the function  $v$ ,

$$v = \chi u. \quad (2.15)$$

Then (2.10), (2.11) and (2.14) imply

$$|A_0 v| \leq C_1 \left[ |\nabla v| + |v| + |\nabla \chi| |\nabla u| + \left( \sum_{|\alpha|=2} |D^\alpha \chi| \right) |u| + |f| \right], \quad \forall x \in G_c, \quad (2.16)$$

$$v|_{\Gamma_c} = \chi g_0, \quad \partial_n v|_{\Gamma_c} = g_0 \partial_n \chi + \chi g_1, \quad (2.17)$$

$$v(x) = 0, x \in G_c \setminus G_{c+\varepsilon}. \quad (2.18)$$

Square both sides of (2.16), multiply by  $\varphi_\lambda^2(x)$  and apply (2.6) ignoring  $C_0$ . We obtain

$$\begin{aligned} & C_1 f^2 \varphi_\lambda^2(x) + C_1 |\nabla \chi|^2 |\nabla u|^2 + C_1 \left( \sum_{|\alpha|=2} |D^\alpha \chi|^2 \right) |u|^2 - \operatorname{div} U \geq \\ & C\lambda \left( 1 - \frac{C_1}{\lambda} \right) (\nabla v)^2 \varphi_\lambda^2(x) + C\lambda^3 \left( 1 - \frac{C_1}{\lambda^3} \right) v^2 \varphi_\lambda^2(x), \forall \lambda > \lambda_0, \forall x \in G_c. \end{aligned}$$

Choose  $\lambda > \lambda_1 := \max(\lambda_0, 2C_1)$  so large that  $C_1/\lambda < 1/2$ . Then with a different constant  $C$

$$\begin{aligned} & C_1 f^2 \varphi_\lambda^2(x) + C_1 |\nabla \chi|^2 |\nabla u|^2 + C_1 \left( \sum_{|\alpha|=2} |D^\alpha \chi|^2 \right) |u|^2 - \operatorname{div} U \geq \\ & C\lambda (\nabla u)^2 \varphi_\lambda^2(x) + C\lambda^3 u^2 \varphi_\lambda^2(x), \forall \lambda > \lambda_1, \forall x \in G_c. \end{aligned}$$

Integrate this inequality over  $G_c$  using Gauss-Ostrogradsky formula as well as (2.1), (2.3), (2.8), (2.14), (2.17) and (2.18). We obtain

$$\begin{aligned} & C_1 e^{2\lambda m} \int_{G_c} f^2 dx + C_1 \lambda^3 e^{2\lambda m} \int_{\Gamma_c} [(\nabla g_0)^2 + g_1^2] dS_x \\ & + C_1 \exp[2\lambda(c+2\varepsilon)] \int_{G_{c+\varepsilon} \setminus G_{c+2\varepsilon}} (|\nabla u|^2 + u^2) dx \\ & \geq \lambda \int_{G_c} (\nabla v)^2 \varphi_\lambda^2 dx + \lambda^3 \int_{G_c} v^2 \varphi_\lambda^2 dx. \end{aligned} \quad (2.19)$$

Since  $G_{c+3\varepsilon} \subset G_{c+2\varepsilon} \subset G_c$ , then strengthening inequality (2.19) and using (2.14), (2.15), we obtain

$$\begin{aligned} & C_1 e^{2\lambda m} \int_{G_c} f^2 dx + C_1 \lambda^3 e^{2\lambda m} \int_{\Gamma_c} [(\nabla g_0)^2 + g_1^2] dS_x + C_1 \exp[2\lambda(c+2\varepsilon)] \int_{G_{c+\varepsilon} \setminus G_{c+2\varepsilon}} (|\nabla u|^2 + u^2) dx \\ & \geq \lambda \int_{G_{c+3\varepsilon}} (\nabla u)^2 \varphi_\lambda^2 dx + \lambda^3 \int_{G_{c+3\varepsilon}} u^2 \varphi_\lambda^2 dx \geq \lambda \exp[2\lambda(c+3\varepsilon)] \int_{G_{c+3\varepsilon}} [(\nabla u)^2 + u^2] dx. \end{aligned}$$

Hence, we have established that

$$\begin{aligned} & C_1 e^{2\lambda m} \int_{G_c} f^2 dx + C_1 \lambda^3 e^{2\lambda m} \int_{\Gamma_c} [(\nabla g_0)^2 + g_1^2] dS_x + C_1 \exp[2\lambda(c+2\varepsilon)] \|u\|_{H^1(G_c)}^2 \\ & \geq \lambda \exp[2\lambda(c+3\varepsilon)] \|u\|_{H^1(G_{c+3\varepsilon})}^2. \end{aligned}$$

Divide both sides of this inequality by  $\lambda \exp[2\lambda(c+3\varepsilon)]$ . Hence, there exists a number

$\lambda_2 = \lambda_2(\varepsilon, m, B, K, G_c) > \lambda_1$  such that

$$\left[ \int_{G_c} f^2 dx + \int_{\Gamma_c} [(\nabla g_0)^2 + g_1^2] dS_x \right] C_1 e^{3\lambda m} + C_1 \exp[-2\lambda\varepsilon] \|u\|_{H^1(G_c)}^2 \geq \|u\|_{H^1(G_{c+3\varepsilon})}^2, \forall \lambda > \lambda_2. \quad (2.20)$$

Using (2.12), we obtain

$$\|u\|_{H^1(G_{c+2\varepsilon})}^2 \leq C_1 \left( \delta^2 e^{3\lambda m} + e^{-2\lambda\varepsilon} \|u\|_{H^1(G_c)}^2 \right). \quad (2.21)$$

We now balance two terms in the right hand side of (2.21) via choosing  $\lambda = \lambda(\delta)$  such that

$$\delta^2 e^{3\lambda m} = e^{-2\lambda \varepsilon}.$$

Hence,

$$\lambda = \ln \left( \delta^{-2(3m+2\varepsilon)^{-1}} \right). \quad (2.22)$$

Hence we should have  $\delta \in (0, \delta_0)$ , where the number  $\delta_0 = \delta_0(\varepsilon, m, B, K, G_c)$  is so small that  $\ln \left( \delta_0^{-2(3m+2\varepsilon)^{-1}} \right) > \lambda_2$ . The target estimate (2.13) follows from (2.21) and (2.22).  $\square$

**Theorem 2.2** (uniqueness). *Let conditions of Theorem 2.1 hold, in (2.11)  $g_0(x) \equiv g_1(x) \equiv 0, x \in \Gamma_c$  and also  $f(x, t) \equiv 0$ . Then  $u(x) \equiv 0$  for  $x \in G_c$ .*

This theorem immediately follows from Theorem 2.1. To prove convergence of the Quasi-Reversibility Method (Section 2.5), we need to replace the pointwise inequality (2.10) with the following integral inequality

$$\int_{G_c} (Au)^2 dx \leq S^2. \quad (2.23)$$

**Theorem 2.3.** *Let the function  $u \in H^2(G_c)$  satisfies inequality (2.23),  $u|_{\Gamma_c} = \partial_n u|_{\Gamma_c} = 0$  and the number  $S \in (0, \delta)$ . Assume that conditions (2.5) hold and that the Carleman estimate of Definition 2.1 is valid. Suppose that there exists a sufficiently small number  $\varepsilon > 0$  such that the domain  $G_{c+3\varepsilon} \neq \emptyset$ . Denote  $m = \max_{\overline{G_c}} \xi(x)$ . Define the number  $\beta = 2\varepsilon/(3m+2\varepsilon) \in (0, 1)$ . Then there exists a sufficiently small number  $\delta_0 = \delta_0(\varepsilon, m, A, G_c) \in (0, 1)$  and a constant  $C_1 = C_1(\varepsilon, m, A, G_c) > 0$  such that if  $\delta \in (0, \delta_0)$ , then the following Hölder stability estimate holds*

$$\|u\|_{H^1(G_{c+3\varepsilon})} \leq C_1 \left( 1 + \|u\|_{H^1(G_c)} \right) \delta^\beta, \forall \delta \in (0, \delta_0).$$

**Proof.** Assume first that the function  $u \in C^2(\overline{G_c})$ . We have

$$S^2 e^{2\lambda m} \geq \int_{G_c} (Au)^2 \varphi_\lambda^2(x) dx \geq \int_{G_c} (A_0 u)^2 \varphi_\lambda^2(x) dx - C_1 \int_{G_c} \left( (\nabla u)^2 + u^2 \right) \varphi_\lambda^2(x) dx.$$

This is equivalent with

$$S^2 e^{2\lambda m} + C_1 \int_{G_c} \left( (\nabla u)^2 + u^2 \right) \varphi_\lambda^2(x) dx \geq \int_{G_c} (A_0 u)^2 \varphi_\lambda^2(x) dx.$$

The rest of the proof is similar with the proof of Theorem 2.1. The replacement of  $u \in C^2(\overline{G_c})$  with  $u \in H^2(G_c)$  can be done via density arguments.  $\square$

**2.3. Derivation of the Carleman estimate for a parabolic operator.** The goal of this Section is to present an example of the derivation of the Carleman estimate. To choose the case of a simplified parabolic operator. Our derivation method is similar with the one of §1 of Chapter 4 of the book of Lavrent'ev, Romanov and Shishatskii [112].

For any  $x \in \mathbb{R}^n$  denote  $y = (x_2, \dots, x_n)$ . Let numbers  $\alpha, \eta \in (0, 1)$  and  $\alpha < \eta$ . Let  $Y, T > 0$  be two arbitrary numbers. Consider the function  $\psi(x, t)$ ,

$$\psi(x, t) = x_1 + \frac{|y|^2}{2Y^2} + \frac{t^2}{2T^2} + \alpha. \quad (2.24)$$

Define the domain  $G_\eta$  as

$$G_\eta = \{(x, t) : \psi(x, t) < \eta, x_1 > 0\} = \left\{ x_1 + \frac{|y|^2}{2Y^2} + \frac{t^2}{2T^2} + \alpha < \eta, x_1 > 0 \right\}.$$



Let  $\lambda, \nu > 1$  be two large parameters which we will choose later. Consider the function  $\varphi_{\lambda, \nu}(x, t)$ ,

$$\varphi(x, t) = \exp(\lambda \psi^{-\nu}). \quad (2.25)$$

$\varphi_{\lambda, \nu}(x, t)$  is the CWF for our parabolic operator  $L$  (below). To simplify notations, we use the notation  $\varphi(x, t)$  instead of  $\varphi_{\lambda, \nu}(x, t)$ . Hence, the boundary of the domain  $G_\eta$  consists of a piece of the hyperplane  $\{x_1 = 0\}$  and a piece of the paraboloid  $\{\psi(x, t) = \eta, x_1 > 0\}$ ,

$$\partial G_\eta = \partial_1 G_\eta \cup \partial_2 G_\eta, \quad (2.26)$$

$$\partial_1 G_\eta = \left\{ x_1 = 0, \frac{|y|^2}{2Y^2} + \frac{t^2}{2T^2} + \alpha < \eta \right\}, \partial_2 G_\eta = \left\{ x_1 > 0, x_1 + \frac{|y|^2}{2Y^2} + \frac{t^2}{2T^2} + \alpha < \eta \right\}. \quad (2.27)$$

Consider a function  $a(x, t)$  for  $(x, t) \in G_\eta$  such that

$$a \in C^1(\overline{G_\eta}), K = \|a\|_{C^1(\overline{G_\eta})}, a(x, t) \geq a_0 = \text{const.} > 0 \text{ for } (x, t) \in G_\eta. \quad (2.28)$$

Consider the parabolic operator  $L$ ,

$$Lu = u_t - a(x, t) \Delta u, (x, t) \in G_\eta. \quad (2.29)$$

By Definition 2.1 we want to estimate now  $(Lu)^2 \varphi^2$  from the below. Introduce the new function  $v = u\varphi$  and express derivatives of the function  $u$  via derivatives of the function  $v$ , using (2.24) and (2.25). Below  $O(1/\lambda), O(1/\nu)$  denote different  $C^1(\overline{G_\eta})$ -functions, which are independent on the function  $u$ , and such that  $|O(1/\lambda)| \leq C/\lambda, |O(1/\nu)| \leq C/\nu, \forall \lambda, \nu \geq 1$ . Here and below in this section  $C = C(a_0, K, G_\eta)$  denotes different positive constants depending only on listed parameters.

**Lemma 2.1.** *Suppose that the function  $a(x, t)$  satisfies conditions (2.28). Then there exist sufficiently large numbers  $\lambda_0 = \lambda_0(a_0, K, G_\eta) > 1, \nu_0 = \nu_0(a_0, K, G_\eta) > 2$  such that for any function  $u \in C^{2,1}(\overline{G_\eta})$  the following estimate holds for all  $\lambda \geq \lambda_0, \nu \geq \nu_0, (x, t) \in G_\eta$*

$$(Lu)^2 \psi^{\nu+2} \varphi^2 \geq -C\lambda\nu (\nabla u)^2 \varphi^2 + C\lambda^3 \nu^4 \psi^{-2\nu-2} \varphi^2 + \text{div } U_1 + \partial_t V_1, \quad (2.30)$$

$$|U_1| + |V_1| \leq C\lambda^3 \nu^3 \psi^{-2\nu-2} \left( (\nabla u)^2 + u^2 \right) \varphi^2. \quad (2.31)$$

**Proof.** We have  $u = v \exp(-\lambda \psi^{-\nu})$ . Hence,

$$\begin{aligned} u_t &= \left( v_t + \frac{t}{T^2} \lambda \nu \psi^{-\nu-1} v \right) \exp(-\lambda \psi^{-\nu}), \\ u_{x_1} &= (v_{x_1} + \lambda \nu \psi^{-\nu-1} v) \exp(-\lambda \psi^{-\nu}), \\ u_{x_1 x_1} &= \left[ v_{x_1 x_1} + 2\lambda \nu \psi^{-\nu-1} v_{x_1} + \lambda^2 \nu^2 \psi^{-2\nu-2} \left( 1 + O\left(\frac{1}{\lambda}\right) \right) v \right] \exp(-\lambda \psi^{-\nu}), \\ u_{x_i} &= \left( v_{x_i} + \frac{x_i}{Y^2} \lambda \nu \psi^{-\nu-1} v \right) \exp(-\lambda \psi^{-\nu}), \quad i \in [2, n], \\ u_{x_i x_i} &= \left[ v_{x_i x_i} + 2\frac{x_i}{Y^2} \lambda \nu \psi^{-\nu-1} v_{x_i} + \lambda^2 \nu^2 \psi^{-2\nu-2} \left( \frac{x_i^2}{Y^4} + O\left(\frac{1}{\lambda}\right) \right) v \right] \exp(-\lambda \psi^{-\nu}), \quad i \in [2, n]. \end{aligned}$$

These equalities imply that

$$\begin{aligned} (Lu)^2 \psi^{\nu+2} \varphi^2 &= \\ \left\{ v_t - a \Delta v - \left[ 2a\lambda \nu \psi^{-\nu-1} v_{x_1} + 2a\lambda \nu \psi^{-\nu-1} \sum_{i=2}^n \frac{x_i}{Y^2} v_{x_i} \right] - a\lambda^2 \nu^2 \psi^{-2\nu-2} \left( \left[ 1 + \sum_{i=2}^n \frac{x_i^2}{Y^4} + O\left(\frac{1}{\lambda}\right) \right] v \right) \right\} \psi^{\nu+2}. \end{aligned}$$



Denote

$$z_1 = v_t, z_2 = -a\Delta v, \quad (2.32)$$

$$z_3 = - \left[ 2a\lambda\nu\psi^{-\nu-1}v_{x_1} + 2a\lambda\nu\psi^{-\nu-1} \sum_{i=2}^n \frac{x_i}{Y^2} v_{x_i} \right], \quad (2.33)$$

$$z_4 = -a\lambda^2\nu^2\psi^{-2\nu-2} \left[ 1 + \sum_{i=2}^n \frac{x_i^2}{Y^4} + O\left(\frac{1}{\lambda}\right) \right] v. \quad (2.34)$$

Hence,

$$(Lu)^2 \psi^{\nu+2} \varphi^2 \geq (z_1^2 + 2z_1z_2 + 2z_1z_3 + z_3^2 + 2z_2z_3 + 2z_3z_4 + 2z_1z_4) \psi^{\nu+2}. \quad (2.35)$$

**Step 1.** Estimate  $2z_1z_2\psi^{\nu+2}$  from the below. By (2.32)

$$\begin{aligned} 2z_1z_2\psi^{\nu+2} &= -2 \sum_{i=1}^n v_t v_{x_i x_i} a\psi^{\nu+2} = \sum_{i=1}^n (-2v_t v_{x_i} a\psi^{\nu+2})_{x_i} \\ &\quad + 2 \sum_{i=1}^n v_{tx_i} v_{x_i} a\psi^{\nu+2} + 2v_t \sum_{i=1}^n v_{x_i} (a\psi^{\nu+2})_{x_i} \\ &= \sum_{i=1}^n (-2v_t v_{x_i} a\psi^{\nu+2})_{x_i} + [(\nabla v)^2 a\psi^{\nu+2}]_t - (\nabla v)^2 (a\psi^{\nu+2})_t + 2v_t \sum_{i=1}^n v_{x_i} (a\psi^{\nu+2})_{x_i}. \end{aligned}$$

Thus, since  $\psi < 1$ , then

$$2z_1z_2\psi^{\nu+2} \geq 2v_t \sum_{i=1}^n v_{x_i} (a\psi^{\nu+2})_{x_i} - C\nu (\nabla v)^2 + \operatorname{div} U_{1,1} + (V_{1,1})_t, \quad (2.36)$$

$$|U_{1,1}| + |V_{1,1}| \leq C\nu (v_t^2 + (\nabla v)^2). \quad (2.37)$$

**Step 2.** Estimate  $(z_1^2 + 2z_1z_2 + 2z_1z_3 + z_3^2) \psi^{\nu+2}$ . Using (2.32), (2.36) and (2.37), we obtain

$$\begin{aligned} &(z_1^2 + 2z_1z_2 + 2z_1z_3 + z_3^2) \psi^{\nu+2} \geq \\ &z_1^2 + z_3^2 + 2z_1 \left( z_3 + \sum_{i=1}^n v_{x_i} (a\psi^{\nu+2})_{x_i} \right) - C\nu (\nabla v)^2 + \operatorname{div} U_{1,1} + (V_{1,1})_t \\ &\geq z_1^2 + z_3^2 - z_1^2 - \left( z_3 + \sum_{i=1}^n v_{x_i} (a\psi^{\nu+2})_{x_i} \right)^2 - C\nu (\nabla v)^2 + \operatorname{div} U_{1,1} + (V_{1,1})_t \\ &= z_3^2 - z_3^2 - 2z_3 \sum_{i=1}^n v_{x_i} (a\psi^{\nu+2})_{x_i} - \left( \sum_{i=1}^n v_{x_i} (a\psi^{\nu+2})_{x_i} \right)^2 - C\nu (\nabla v)^2 + \operatorname{div} U_{1,1} + (V_{1,1})_t. \end{aligned}$$

Using (2.33), we obtain

$$-2z_3 \sum_{i=1}^n v_{x_i} (a\psi^{\nu+2})_{x_i} = 4a^2\lambda\nu(\nu+2) \left[ v_{x_1} + \sum_{i=2}^n \frac{x_i}{Y^2} v_{x_i} \right]^2 - C\lambda\nu (\nabla v)^2 \geq -C\lambda\nu (\nabla v)^2.$$

Thus,

$$(z_1^2 + 2z_1z_2 + 2z_1z_3 + z_3^2) \psi^{\nu+2} \geq -C\lambda\nu (\nabla v)^2 + \operatorname{div} U_{1,1} + (V_{1,1})_t. \quad (2.38)$$

**Step 3.** Estimate  $2z_2z_3\psi^{\nu+2}$ ,

$$2z_2z_3\psi^{\nu+2} = 4a^2\lambda\nu\psi \left( v_{x_1} + \sum_{i=2}^n \frac{x_i}{Y^2} v_{x_i} \right) \Delta v. \quad (2.39)$$

Consider for example the term  $4a^2\lambda\nu\psi v_{x_1} \Delta v$ ,

$$\begin{aligned} 4a^2\lambda\nu\psi v_{x_1} \Delta v &= 4\lambda\nu \sum_{i=1}^n v_{x_i x_i} v_{x_1} a^2 \psi = \sum_{i=1}^n (4\lambda\nu a^2 \nu_{x_i} v_{x_1} \psi)_{x_i} \\ &\quad - \sum_{i=1}^n 4\lambda\nu a^2 \psi \nu_{x_i} v_{x_i x_1} - \sum_{i=1}^n 4\lambda\nu (a^2 \psi)_{x_i} \nu_{x_i} v_{x_1} \\ &\geq \sum_{i=1}^n (4\lambda\nu a^2 \nu_{x_i} v_{x_1} \psi)_{x_i} + \left( \sum_{i=1}^n 2\lambda\nu a^2 \psi \nu_{x_i}^2 \right)_{x_1} - 2\lambda\nu (a^2 \psi)_{x_1} (\nabla v)^2 - C\lambda\nu (\nabla v)^2 \\ &\geq -C\lambda\nu (\nabla v)^2 + \operatorname{div} U_{1,2}. \end{aligned}$$

Thus,  $4a^2\lambda\nu\psi v_{x_1} \Delta v \geq -C\lambda\nu (\nabla v)^2 + \operatorname{div} U_{1,2}$ . Similarly we obtain using (2.39)

$$2z_2z_3\psi^{\nu+2} \geq -C\lambda\nu (\nabla v)^2 + \operatorname{div} U_{1,3}. \quad (2.40)$$

**Step 4.** Estimate  $2z_3z_4\psi^{\nu+2}$ ,

$$\begin{aligned} 2z_3z_4\psi^{\nu+2} &= 4a^2\lambda^3\nu^3\psi^{-2\nu-1} \left[ 1 + \sum_{i=2}^n \frac{x_i^2}{Y^4} + O\left(\frac{1}{\lambda}\right) \right] \left[ v_{x_1} + \sum_{i=2}^n \frac{x_i}{Y^2} v_{x_i} \right] v \\ &= \left[ 2a^2\lambda^3\nu^3\psi^{-2\nu-1} \left( 1 + \sum_{i=2}^n \frac{x_i^2}{Y^4} + O\left(\frac{1}{\lambda}\right) \right) v^2 \right]_{x_1} \\ &\quad + \left[ 2a^2\lambda^3\nu^3\psi^{-2\nu-1} \left( 1 + \sum_{j=2}^n \frac{x_j^2}{Y^4} + O\left(\frac{1}{\lambda}\right) \right) \left( \sum_{i=2}^n \frac{x_i}{Y^2} \right) v^2 \right]_{x_i} \\ &\quad + 2a^2\lambda^3\nu^3 (2\nu+1) \psi^{-2\nu-2} \left[ 1 + \left( \sum_{i=2}^n \frac{x_i}{Y^2} \right)^2 + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{\nu}\right) \right] v^2. \end{aligned}$$

Thus,

$$2z_3z_4\psi^{\nu+2} \geq C\lambda^3\nu^4\psi^{-2\nu-2}v^2 + \operatorname{div} U_{1,4}. \quad (2.41)$$

Similarly,

$$2z_1z_4\psi^{\nu+2} \geq -C\lambda^2\nu^3\psi^{-\nu-1}v^2 + \operatorname{div} U_{1,5} + (V_{1,2})_t. \quad (2.42)$$

Summing up (2.38)-(2.42) and replacing  $v$  with  $u = v\varphi^{-1}$ , we obtain (2.30), (2.31).  $\square$

As one can see that we have one positive and one negative term in the right hand side in estimate (2.30). Therefore, we need to balance them somehow to obtain only positive terms. To do this, we prove Lemma 2.2 first.

**Lemma 2.2.** *Suppose that the function  $a(x, t)$  satisfies conditions (2.28). Then there exist sufficiently large numbers  $\lambda_0 = \lambda_0(a_0, K, G_\eta) > 1, \nu_0 = \nu_0(a_0, K, G_\eta) > 2$  such that for any function  $u \in C^{2,1}(\overline{G}_\eta)$  the following estimate holds for all  $\lambda \geq \lambda_0, \nu \geq \nu_0, (x, t) \in G_\eta$*

$$(u_t - a(x, t) \Delta u) u \varphi^2 \geq C (\nabla u)^2 \varphi^2 - C\lambda^2\nu^2\psi^{-2\nu-2}\varphi^2 u^2 + \operatorname{div} U_2 + \partial_t V_2, \quad (2.43)$$

$$|U_2| + |V_2| \leq C \left( |\nabla u|^2 + u^2 \right) \varphi^2. \quad (2.44)$$

**Proof.** We have

$$\begin{aligned}
(u_t - a(x, t) \Delta u) u \varphi^2 &= \left( \frac{1}{2} u^2 \varphi^2 \right)_t + \frac{t}{2T^2} \lambda \nu \psi^{-\nu-1} \varphi^2 u^2 + \sum_{i=1}^n (-a(x, t) u_{x_i} u \varphi^2)_{x_i} + a(\nabla u)^2 \varphi^2 \\
&\quad - 2a\lambda \nu \sum_{i=1}^n \psi_{x_i} \psi^{-\nu-1} u_{x_i} u \varphi^2 + \sum_{i=1}^n a_{x_i} u_{x_i} u \varphi^2 \\
&\geq C(\nabla u)^2 \varphi^2 - C\lambda^2 \nu^2 \psi^{-2\nu-2} \varphi^2 u^2 + \operatorname{div} U_2 + V_{2t}. \quad \square
\end{aligned}$$

**Theorem 2.4.** Suppose that the function  $a(x, t)$  satisfies conditions (2.28). Then there exist sufficiently large numbers  $\lambda_0 = \lambda_0(a_0, K, G_\eta) > 1, \nu_0 = \nu_0(a_0, K, G_\eta) > 2$  such that for any function  $u \in C^{2,1}(\overline{G}_\eta)$  the following Carleman estimate holds for all  $\lambda \geq \lambda_0, \nu \geq \nu_0, (x, t) \in G_\eta$

$$(Lu)^2 \varphi^2 \geq C\lambda \nu (\nabla u)^2 \varphi^2 + C\lambda^3 \nu^4 \psi^{-2\nu-2} \varphi^2 u^2 + \operatorname{div} U + V_t, \quad (2.45)$$

$$|U| + |V| \leq C\lambda^3 \nu^3 \psi^{-2\nu-2} \left( (\nabla u)^2 + u^2 \right) \varphi^2. \quad (2.46)$$

**Proof.** Multiply (2.43) and (2.44) by  $2\lambda\nu$  and sum up with (2.30), (2.31). We obtain

$$(Lu)^2 \psi^{\nu+2} \varphi^2 + 2C\lambda \nu (Lu) u \varphi^2 \geq C\lambda \nu (\nabla u)^2 \varphi^2 + C\lambda^3 \nu^4 \psi^{-2\nu-2} \left( 1 - \frac{1}{\nu} \right) \varphi^2 u^2 + \operatorname{div} U + V_t, \quad (2.47)$$

where the vector function  $(U, V)$  satisfies (2.46). Choose  $\nu_0 = \nu_0(a_0, K, G_\eta) > 2$ . Also, since  $\psi^{\nu+2} < 1$ , we have

$$(Lu)^2 \psi^{\nu+2} \varphi^2 + 2\lambda \nu (Lu) u \varphi^2 \leq 2(Lu)^2 \varphi^2 + \lambda^2 \nu^2 \varphi^2 u^2.$$

Combining this with (2.47), we obtain (2.45).  $\square$

We now want to incorporate higher order derivatives  $u_t, u_{x_i x_j}$  in the Carleman estimate of Theorem 2.3. To do this, we prove Lemma 2.4 and Theorem 2.4.

**Lemma 2.4.** Suppose that the function  $a(x, t)$  satisfies conditions (2.28). Fix the number  $\nu := \nu_0(a_0, K, G_\eta) > 2$  of Theorem 2.3. There exists sufficiently large number  $\lambda_0 = \lambda_0(a_0, K, G_\eta) > 1$  such that for any function  $u \in C^3(\overline{G}_\eta)$  the following estimate holds for all  $\lambda \geq \lambda_0, (x, t) \in G_\eta$

$$(Lu)^2 \varphi^2 \geq \frac{1}{2} \left( u_t^2 + \sum_{i,j=1}^n u_{x_i x_j}^2 \right) \varphi^2 - C\lambda^2 (\nabla u)^2 \varphi^2 + \operatorname{div} U_3 + \partial_t V_3, \quad (2.48)$$

$$|U_3| + |V_3| \leq C \left( u_t^2 + \sum_{i,j=1}^n u_{x_i x_j}^2 + (\nabla u)^2 \right) \varphi^2. \quad (2.49)$$

**Proof.** We have

$$(Lu)^2 \varphi^2 \geq \left( u_t^2 + 2au_t \Delta u + a_0^2 (\Delta u)^2 \right) \varphi^2. \quad (2.50)$$

**Step 1.** Estimate  $2au_t \Delta u \varphi^2$ ,

$$\begin{aligned}
2au_t \Delta u \varphi^2 &= \sum_{i=1}^n 2au_t u_{x_i x_i} \varphi^2 = \sum_{i=1}^n (2au_t u_{x_i} \varphi^2)_{x_i} - 2 \sum_{i=1}^n au_{tx_i} u_{x_i} \varphi^2 \\
&\quad - 2 \sum_{i=1}^n a_{x_i} u_t u_{x_i} \varphi^2 - 4\lambda \nu \psi^{-\nu-1} (\psi)_{x_i} \sum_{i=1}^n au_t u_{x_i} \varphi^2.
\end{aligned}$$

We have

$$-2 \sum_{i=1}^n a u_{t x_i} u_{x_i} \varphi^2 = \left( -a (\nabla u)^2 \varphi^2 \right)_t - 4\lambda \nu \frac{t}{T^2} \psi^{-\nu-1} a (\nabla u)^2 \varphi^2 + a_t (\nabla u)^2 \varphi^2.$$

Since the number  $\nu := \nu_0(a_0, K, G_\eta)$  depends on the same parameters as the constant  $C$ , we can incorporate  $\nu$  in  $C$ . Hence, we obtain

$$2a u_t \Delta u \varphi^2 \geq -u_t^2 \varphi^2 - C\lambda^2 (\nabla u)^2 \varphi^2 + \operatorname{div} U_{3,1} + (V_{3,1})_t. \quad (2.51)$$

**Step 2.** Estimate  $(\Delta u)^2 \varphi^2$ ,

$$\begin{aligned} (\Delta u)^2 \varphi^2 &= \sum_{i,j=1}^n u_{x_i x_i} u_{x_j x_j} \varphi^2 = \sum_{j=1}^n \left( \sum_{i=1}^n u_{x_i x_i} u_{x_j} \varphi^2 \right)_{x_j} \\ &\quad - \sum_{i=1}^n \sum_{j=1}^n u_{x_i x_i x_j} u_{x_j} \varphi^2 + 2\lambda \nu \psi^{-\nu-1} \sum_{i,j=1}^n u_{x_i x_i} u_{x_j} \psi_{x_j} \varphi^2 \\ &\geq \sum_{i=1}^n \left( - \sum_{j=1}^n u_{x_i x_j} u_{x_j} \varphi^2 \right)_{x_i} + \frac{1}{2} \sum_{i,j=1}^n u_{x_i x_j}^2 \varphi^2 - C\lambda^2 (\nabla u)^2 \varphi^2 \\ &\quad + \sum_{j=1}^n \left( \sum_{i=1}^n u_{x_i x_i} u_{x_j} \varphi^2 \right)_{x_j} = \frac{1}{2} \sum_{i,j=1}^n u_{x_i x_j}^2 \varphi^2 - C\lambda^2 (\nabla u)^2 \varphi^2 + \operatorname{div} U_{3,2}, \end{aligned}$$

Hence, we have obtained that

$$(\Delta u)^2 \varphi^2 \geq \frac{1}{2} \sum_{i,j=1}^n u_{x_i x_j}^2 \varphi^2 - C\lambda^2 (\nabla u)^2 \varphi^2 + \operatorname{div} U_{3,2}, \quad (2.52)$$

$$|U_{3,2}| \leq C \left( \sum_{i,j=1}^n u_{x_i x_j}^2 + (\nabla u)^2 \right) \varphi^2. \quad (2.53)$$

Comparing (2.50), (2.51), (2.52) and (2.53), we obtain (2.48) and (2.49).  $\square$

**Theorem 2.5.** Suppose that the function  $a(x, t)$  satisfies conditions (2.28). Fix the number  $\nu := \nu_0(a_0, K, G_\eta) > 2$  of Theorem 2.3. Then there exists a sufficiently large number  $\lambda_0 = \lambda_0(a_0, K, G_\eta) > 1$  such that for any function  $u \in C^3(\overline{G}_\eta)$  the following estimate holds for all  $\lambda \geq \lambda_0, (x, t) \in G_\eta$

$$(Lu)^2 \varphi^2 \geq \frac{C_0}{\lambda} \left( u_t^2 + \sum_{i,j=1}^n u_{x_i x_j}^2 \right) \varphi^2 + C\lambda (\nabla u)^2 \varphi^2 + C\lambda^3 u^2 \varphi^2 + \operatorname{div} U + V_t, \quad (2.54)$$

$$|U| + |V| \leq \frac{C_0}{\lambda} \left( u_t^2 + \sum_{i,j=1}^n u_{x_i x_j}^2 \right) \varphi^2 + C\lambda^3 \left( (\nabla u)^2 + u^2 \right) \varphi^2, \quad (2.55)$$

where the constant  $C_0 = C_0(a_0, K, G_\eta) > 0$  depends on the same parameters as the constant  $C$  above.

**Proof.** Divide inequality (2.48) by  $b\lambda$  with an appropriate constant  $b = b(a_0, K, G_\eta) > 1$ . Next, add the resulting inequality to (2.45), where set  $\nu := \nu_0$ . Then we obtain (2.54), (2.55).  $\square$

**Remark 2.2.** If one would set above  $u_t \equiv 0$  and would ignore the term  $t^2/T^2$  in the function  $\psi$ , then one would obtain analogs of these results for the general elliptic operator. A close analog of Theorem 2.4 is valid for a general parabolic operator of the second order, see §1 of Chapter 4 of the book of Lavrent'ev, Romanov and Shishatskii [112]. Therefore, Theorems 2.1-2.3 are valid for this operator.

**2.4. Carleman estimate for a hyperbolic operator.** Theorem 2.5 is proven in the book of Beilina and Klivanov [14]. Similar theorems were established in earlier books of Isakov [65, 67] and Klivanov and Timonov [86]. Theorem 2.6 can be found in §4 of Chapter 4 of the book of Lavrent'ev, Romanov and Shishatskii [112]. We do not reproduce proofs here for brevity.

For brevity we consider a simple domain  $\Omega = \{|x| < R\} \subset \mathbb{R}^n$ . Let  $T = \text{const.} > 0$ . Denote

$$Q_T = \Omega \times (0, T), Q_T^\pm = \Omega \times (-T, T), S_T = \partial\Omega \times (0, T), S_T^\pm = \partial\Omega \times (-T, T).$$

Choose a point  $x_0 \in \mathbb{R}^n$ . In particular, we can have  $x_0 \in \Omega$ . Let the number  $\eta \in (0, 1)$ . Let  $\lambda > 1$  be a large parameter. Define functions  $\xi(x, t), \varphi(x, t)$  as

$$\xi(x, t) = |x - x_0|^2 - \eta t^2, \varphi(x, t) = \exp[\lambda \xi(x, t)]. \quad (2.56)$$

For a number  $\gamma > 0$  define the domain  $G_\gamma$  as

$$G_\gamma = \{\xi(x, t) > \gamma\}. \quad (2.57)$$

One can choose  $\gamma$  such that

$$G_\gamma \subset Q_T^\pm. \quad (2.58)$$

**Theorem 2.5.** *Let  $\Omega = \{|x| < R\} \subset \mathbb{R}^n, n \geq 2$ . Let conditions (2.56), (2.59) and (2.60) be in place. Let  $d = \text{const.} \geq 1$ . Let the function  $c(x)$  satisfies the following conditions*

$$c^{-2}(x) \in [1, d], \forall x \in \overline{\Omega}, c \in C^1(\overline{\Omega}), \quad (2.59)$$

$$(x - x_0, \nabla c^{-2}(x)) \geq 0, \forall x \in \overline{\Omega}, \quad (2.60)$$

for a certain point  $x_0 \in \Omega$ , where  $(\cdot, \cdot)$  denotes the scalar product in  $\mathbb{R}^n$ . Let  $Lu = u_{tt} - c^2(x) \Delta u$  be a hyperbolic operator. Let

$$P = P(x_0, \Omega) = \max_{x \in \overline{\Omega}} |x - x_0|. \quad (2.61)$$

Then there exists a number  $\eta_0 = \eta_0(\Omega, d, P, \|\nabla c\|_{C(\overline{\Omega})}) \in (0, 1)$  such that for any  $\eta \in (0, \eta_0)$  one can choose a sufficiently large number  $\lambda_0 = \lambda_0(d, P, \|\nabla c\|_{C(\overline{\Omega})}, \eta, \gamma) > 1$  and the number  $C = C(d, P, \|\nabla c\|_{C(\overline{\Omega})}, \eta, \gamma) > 0$ , such that for all  $u \in C^2(\overline{G_\gamma})$  and for all  $\lambda \geq \lambda_0$  the following pointwise Carleman estimate holds

$$(Lu)^2 \varphi^2 \geq C\lambda \left( |\nabla u|^2 + u_t^2 \right) \varphi^2 + \lambda^3 u^2 \varphi^2 + \text{div } U + V_t, \text{ in } G_\gamma,$$

where

$$|U| \leq C\lambda^3 \left( |\nabla u|^2 + u_t^2 + u^2 \right) \varphi^2, \quad (2.62)$$

$$|V| \leq C\lambda^3 \left[ |t| \left( u_t^2 + |\nabla u|^2 + u^2 \right) + (|\nabla u| + |u|) |u_t| \right] \varphi^2. \quad (2.63)$$

In particular, (2.63) implies that if either  $u(x, 0) = 0$  or  $u_t(x, 0) = 0$ , then

$$V(x, 0) = 0. \quad (2.64)$$

**Theorem 2.6.** *Let  $c(x) \equiv 1$ . Then Theorem 2.5 is valid for any  $\eta \in (0, 1)$ .*

**2.5. The Quasi-Reversibility Method (QRM).** In this section we use notations of Sections 2.1, 2.2. Let  $A$  be the operator of Section 2.1. QRM delivers an approximate solution of the following Cauchy problem

$$Au = f, x \in G_c, \quad (2.65)$$

$$f \in L_2(G_c), u \in H_0^2(G_c) = \{u \in H^2(G_c) : u|_{\Gamma_c} = \partial_n u|_{\Gamma_c} = 0\}. \quad (2.66)$$

To find that approximate solution, QRM minimizes the following Tikhonov functional with the regularization parameter  $\gamma$

$$J_\gamma(u) = \|Au - f\|_{L_2(G_c)}^2 + \gamma \|u\|_{H_0^2(G_c)}^2, u \in H_0^2(G_c). \quad (2.67)$$

The variational principle implies that any minimizer  $u_\gamma \in H_0^2(G_c)$  satisfies the following integral identity

$$(Au_\gamma, Av) + \gamma [u, v] = (f, Av), \forall v \in H_0^2(G_c), \quad (2.68)$$

where  $(\cdot, \cdot)$  and  $[\cdot, \cdot]$  are scalar products in  $L_2(G_c)$  and  $H^2(G_c)$  respectively. Riesz theorem and (2.68) imply Lemma 2.5.

**Lemma 2.5.** *Let  $A$  be the operator defined in (2.4), (2.5) and the function  $f \in L_2(G_c)$ . Then for any  $\gamma > 0$  there exists unique minimizer  $u \in H_0^2(G_c)$  of the functional (2.67). Furthermore, with a constant  $\overline{C} = \overline{C}(A)$  the following estimate holds  $\|u_\gamma\|_{H^2(G_c)} \leq \overline{C}/\sqrt{\gamma}$ .*

**Theorem 2.7** (convergence). *Assume that conditions (2.5) hold and that the Carleman estimate of Definition 2.1 is valid. Suppose that there exists a sufficiently small number  $\varepsilon > 0$  such that the domain  $G_{c+3\varepsilon} \neq \emptyset$ . Let the function  $u^*$  be the exact solution of the problem (2.65), (2.66) with the exact function  $f^* \in L_2(G_c)$ . Let  $\|f - f^*\|_{L_2(G_c)} \leq \delta$  and  $\gamma = \delta^2$ . Denote  $m = \max_{\overline{G_c}} \xi(x)$ . Define the number  $\beta = 2\varepsilon/(3m + 2\varepsilon) \in (0, 1)$ . There exists a sufficiently small number  $\delta_0 = \delta_0(\varepsilon, m, A, G_c, \|u^*\|_{H^2(G_c)}) \in (0, 1)$  and a constant  $C_1 = C_1(\varepsilon, m, A, G_c) > 0$  such that if  $\delta \in (0, \delta_0)$ , then the following convergence rate is valid*

$$\|u_\gamma - u^*\|_{H^1(G_{c+3\varepsilon})} \leq C_1 \left(1 + \|u^*\|_{H^2(G_c)}\right) \delta^{\beta/2}, \forall \delta \in (0, \delta_0).$$

**Proof.** We have

$$(Au^*, Av) + \gamma [u^*, v] = (f^*, Av) + \gamma [u^*, v], \forall v \in H_0^2(G_c).$$

Subtract this identity from (2.68) and denote  $w_\gamma = u_\gamma - u^*$ ,  $g = f - f^*$ . We obtain

$$(Aw_\gamma, Av) + \gamma [w_\gamma, v] = (g, Av) - \gamma [u^*, v], \forall v \in H_0^2(G_c). \quad (2.69)$$

Setting in (2.69)  $v := w_\gamma$  and using Cauchy-Bunyakovsky inequality, we obtain

$$\begin{aligned} \|Aw_\gamma\|_{L_2(G_c)}^2 &\leq \left(1 + \|u^*\|_{H^2(G_c)}^2\right) \delta^2 \leq \delta, \\ \|w_\gamma\|_{H^2(G_c)}^2 &\leq 1 + \|u^*\|_{H^2(G_c)}^2. \end{aligned}$$

The rest of the proof follows from Theorem 2.3.  $\square$

**2.6. Published results about QRM.** As it is clear from (2.67), QRM is a special form of the Tikhonov regularization functional, see [5, 14, 49, 145] for the theory of this functional. In “conventional” Tikhonov functional the originating operator is continuous. On the other hand in QRM the originating operator is a PDE operator, which is continuous only if its domain is  $H^2$ . QRM is well suitable for providing approximate solutions for ill-posed Cauchy problems for PDEs, including boundary value problems with over-determined boundary conditions. QRM was first introduced by Lattes and Lions in their book [111]. This book shows how to apply QRM to ill-posed Cauchy problems for all three main types of PDE operators of the second

order: elliptic, parabolic and hyperbolic. Although convergence theorems were proven in [111], convergence rates for QRM were not established there.

The first work where a Carleman estimate was applied to get convergence rate of QRM was one of Klivanov and Santosa [80]. In [80] QRM was applied to the Cauchy problem for the Laplace equation. In Chapter 2 of the book of Klivanov and Timonov [86] Carleman estimates were also applied to establish convergence rate of QRM for ill-posed Cauchy problems for elliptic, parabolic and hyperbolic PDEs. Next, Carleman estimates were used to prove convergence rate of QRM for the Cauchy problem for the Laplace equation by Bourgeois [32], Bourgeois and Darde [33] and Cao, Klivanov and Pereverzev [41]. The QRM for the problem of determining of the initial condition in the parabolic PDE from boundary measurements was considered by the author in [89, 95]. Papers [33, 41, 80] contain numerical results. As to the application of QRM to the problem with the lateral Cauchy data for the hyperbolic PDE, see Sections 5.4 and 5.5.

While above citations of QRM are concerned only with linear problems, it was recently applied by the author with coauthors to solve MCIPs with backscattering data via the approximately globally convergent method [99, 100, 101, 102, 103], also see chapter 6 of [14]. The main difference between the latter application of the QRM and the conventional one is that MCIPs are nonlinear.

### 3. The Bukhgeim-Klivanov Method.

**3.1. Estimating an integral.** First, we estimate a Volterra-like integral with a weight function. For the first time an analog of Lemma 3.1 was proven by the author in [74]. Next, that proof was published in some of above cited follow up papers of the author about BK. The estimate of this lemma with the parameter  $1/\lambda$  in it was first published in the book of Klivanov and Timonov [86], also see Section 1.10.3 in the book [14].

**Lemma 3.1.** *Let the function  $\varphi \in C^1[0, a]$  and  $\varphi'(t) \leq -b$  in  $[0, a]$ , where  $b = \text{const} > 0$ . For a function  $g \in L_2(-a, a)$  consider the integral*

$$I(g, \lambda) = \int_{-a}^a \left( \int_0^t g(\tau) d\tau \right)^2 \exp[2\lambda\varphi(t^2)] dt, \lambda = \text{const.} > 0.$$

Then,

$$I(g, \lambda) \leq \frac{1}{4\lambda b} \int_{-a}^a g^2(t) \exp[2\lambda\varphi(t^2)] dt.$$

**Proof.** We have for  $t > 0$

$$\begin{aligned} t \exp[2\lambda\varphi(t^2)] &= t \frac{4\lambda\varphi'(t^2)}{4\lambda\varphi'(t^2)} \exp[2\lambda\varphi(t^2)] \\ &= \frac{1}{4\lambda\varphi'(t^2)} \frac{d}{dt} \{ \exp[2\lambda\varphi(t^2)] \} = -\frac{1}{4\lambda\varphi'(t^2)} \frac{d}{dt} \{ -\exp[2\lambda\varphi(t^2)] \} \\ &\leq \frac{1}{4\lambda b} \frac{d}{dt} \{ -\exp[2\lambda\varphi(t^2)] \}. \end{aligned}$$

Hence,

$$\begin{aligned} \int_0^a \left( \int_0^t g(\tau) d\tau \right)^2 \exp(2\lambda\varphi(t^2)) dt &\leq \int_0^a \exp(2\lambda\varphi(t^2)) t \left( \int_0^t g^2(\tau) d\tau \right) dt \\ &\leq \frac{1}{4\lambda b} \int_0^a \frac{d}{dt} [-\exp(2\lambda\varphi(t^2))] \left( \int_0^t g^2(\tau) d\tau \right) dt \end{aligned}$$



$$\begin{aligned}
&= -\frac{1}{4\lambda b} \exp(2\lambda\varphi(a^2)) \int_0^a g^2(\tau) d\tau + \frac{1}{4\lambda b} \int_0^a g^2(\tau) \exp(2\lambda\varphi(t^2)) dt \\
&\leq \frac{1}{4\lambda b} \int_0^a g^2(\tau) \exp(2\lambda\varphi(t^2)) dt.
\end{aligned}$$

Thus, we have proved that

$$\int_0^a \exp(2\lambda\varphi(t^2)) \left( \int_0^t g(\tau) d\tau \right)^2 dt \leq \frac{1}{4\lambda b} \int_0^a g^2(\tau) \exp(2\lambda\varphi(t^2)) dt.$$

Similarly,

$$\int_{-a}^0 \exp(2\lambda\varphi(t^2)) \left( \int_0^t g(\tau) d\tau \right)^2 dt \leq \frac{1}{4\lambda b} \int_{-a}^0 g^2(\tau) \exp(2\lambda\varphi(t^2)) dt. \quad \square$$

**3.2. An MCIP for a hyperbolic equation.** In this section we use notations of Section 2.4. Let functions  $a_\alpha(x, t) \in C(\overline{Q}_T)$ ,  $|\alpha| \leq 1$  and the function  $c(x) \in C^1(\overline{\Omega})$ ,  $c(x) \geq \text{const} > 0$ . Let the function  $u \in C^2(\overline{Q}_T)$  be the solution of the following initial boundary value problem

$$c(x) u_{tt} = \Delta u + \sum_{|\alpha| \leq 1} a_\alpha(x) D_x^\alpha u, \quad \text{in } Q_T, \quad (3.1)$$

$$u(x, 0) = f_0(x), \quad u_t(x, 0) = f_1(x), \quad (3.2)$$

$$u|_{S_T} = p(x, t), \quad S_T = \partial\Omega \times (0, T). \quad (3.3)$$

**Coefficient Inverse Problem for the Hyperbolic Equation (3.1).** Let the Neumann boundary condition be known,

$$\frac{\partial u}{\partial n}|_{S_T} = q(x, t). \quad (3.4)$$

Determine one of  $x$ -dependent coefficients of equation (3.1), assuming that other coefficients are known, so as and functions  $f_0, f_1, p, q$  in (3.2)-(3.4).

**Theorem 3.1.** *Let the domain  $\Omega = \{|x| < R\} \subset \mathbb{R}^n$ ,  $n \geq 2$ . Denote  $b(x) = 1/\sqrt{c(x)}$ . Let the function  $b(x)$  satisfies conditions (2.59), (2.60). In addition, let coefficients  $a_\alpha \in C(\overline{\Omega})$ . Consider two cases:*

**Case 1.** *The coefficient  $c(x)$  is unknown and all other coefficients  $a_\alpha(x, t)$  are known. In this case we assume that*

$$\Delta f_0(x) + \sum_{|\alpha| \leq 1} a_\alpha(x) D_x^\alpha f_0(x) \neq 0 \quad \text{for } x \in \overline{\Omega}. \quad (3.5)$$

*Then for a sufficiently large  $T > 0$  there exists at most one pair of functions  $(u, c)$  satisfying (3.1)-(3.4) and such that  $u \in C^4(\overline{Q}_T)$ .*

**Case 2.** *Let  $\alpha_0$  be one of multi-indices in (3.1),  $|\alpha_0| \leq 1$ . Let the coefficient  $a_{\alpha_0}(x)$  be unknown and all other coefficients are known. In this case we assume that*

$$D_x^{\alpha_0} f_0(x) \neq 0 \quad \text{for } x \in \overline{\Omega}. \quad (3.6)$$

*Then for a sufficiently large  $T > 0$  there exists at most one pair of functions  $(u, a_{\alpha_0})$  satisfying (3.1)-(3.4) and such that  $u \in C^{3+|\alpha_0|}(\overline{Q}_T)$ .*

*If in (3.2)  $f_0(x) \equiv 0$ , then conditions of these two cases should be imposed on the function  $f_1(x)$ , the required smoothness of the function  $u$  should be  $u \in C^5(\overline{Q}_T)$  in Case 1 and  $u \in C^{4+|\alpha_0|}(\overline{Q}_T)$  in Case 2, and the above statements about uniqueness would still hold.*

**Proof.** First, we note that if  $f_0(x) \equiv 0$ , then one should consider in this proof  $u_t$  instead of  $u$ , and the rest of the proof is the same as the one below. We prove this theorem only for Case 1, since Case 2 is similar. Assume that there exist two solutions  $(u_1, c_1)$  and  $(u_2, c_2)$ . Denote  $\tilde{u} = u_1 - u_2, \tilde{c} = c_1 - c_2$ . Since

$$c_1 u_{1tt} - c_2 u_{2tt} = c_1 u_{1tt} - c_1 u_{2tt} + (c_1 - c_2) u_{2tt} = c_1 \tilde{u}_{tt} + \tilde{c} u_{2tt},$$

then (3.1)-(3.4) lead to

$$L\tilde{u} = c_1(x) \tilde{u}_{tt} - \Delta \tilde{u} - \sum_{j=1}^n a_\alpha(x) D_x^\alpha \tilde{u} = \tilde{c}(x) B(x, t), \text{ in } Q_T, \quad (3.7)$$

$$\tilde{u}(x, 0) = 0, \tilde{u}_t(x, 0) = 0, \quad (3.8)$$

$$\tilde{u}|_{S_T} = \frac{\partial \tilde{u}}{\partial n}|_{S_T} = 0, \quad (3.9)$$

$$B(x, t) := -u_{2tt}(x, t). \quad (3.10)$$

Setting in (3.1)  $c := c_2, u := u_2, t := 0$  and using (3.5) and (3.9), we obtain

$$B(x, 0) = -c_2^{-1}(x) \left( \Delta f_0(x) + \sum_{|\alpha| \leq 1} a_\alpha(x) D_x^\alpha f_0(x) \right) \neq 0 \text{ for } x \in \overline{\Omega}.$$

Hence, there exists a sufficiently small positive number  $\varepsilon$ , such that

$$B(x, t) \neq 0 \text{ in } \overline{Q}_\varepsilon = \overline{\Omega} \times [0, \varepsilon]. \quad (3.11)$$

By (3.7)

$$\tilde{c}(x) = \frac{L\tilde{u}}{B(x, t)} \text{ in } \overline{Q}_\varepsilon.$$

Hence,

$$\frac{\partial}{\partial t} [\tilde{c}(x)] = \frac{\partial}{\partial t} \left[ \frac{L\tilde{u}}{B(x, t)} \right] = 0 \text{ in } \overline{Q}_\varepsilon.$$

Or

$$L\tilde{u}_t = \frac{B_t}{B} (L\tilde{u}) \text{ in } \overline{Q}_\varepsilon. \quad (3.12)$$

Denote

$$h(x, t) = \frac{B_t}{B}(x, t). \quad (3.13)$$

By (3.11) and (3.13)

$$h \in C^2(\overline{Q}_\varepsilon). \quad (3.14)$$

Denote

$$v(x, t) = \tilde{u}_t(x, t) - h\tilde{u}(x, t). \quad (3.15)$$

We can consider (3.15) as an ordinary differential equation with respect to  $\tilde{u}(x, t)$  with the initial condition from (3.8), i.e.  $\tilde{u}(x, 0) = 0$ . Hence, using (3.13), (3.14) and (3.15), we obtain

$$\tilde{u}(x, t) = \int_0^t K(x, t, \tau) v(x, \tau) d\tau, \quad (3.16)$$

$$K(x, t, \tau) = \frac{B(x, t)}{B(x, \tau)} \in C^2(\overline{\Omega} \times [0, \varepsilon] \times [0, \varepsilon]), \quad (3.17)$$

$$v(x, 0) = 0. \quad (3.18)$$

Using (3.13), (3.14), (3.15), (3.16) and (3.17), we obtain in  $\overline{Q}_\varepsilon$

$$\begin{aligned} c_1(\tilde{u}_t)_{tt} - hc_1\tilde{u}_{tt} &= c_1v_{tt} + 2c_1h_t\tilde{u}_t + c_1h_{tt}\tilde{u} \\ &= c_1v_{tt} + 2c_1h_tv + 2c_1h_t \int_0^t K_t(x, t, \tau) v(x, \tau) d\tau + c_1h_{tt} \int_0^t K(x, t, \tau) v(x, \tau) d\tau, \end{aligned}$$

$$\begin{aligned} \Delta\tilde{u}_t - h\Delta\tilde{u} &= \Delta v + 2\nabla h \nabla \tilde{u} + \Delta h \tilde{u} \\ &= \Delta v + 2\nabla h \nabla \left( \int_0^t K(x, t, \tau) v(x, \tau) d\tau \right) + \Delta h \int_0^t K(x, t, \tau) v(x, \tau) d\tau. \end{aligned}$$

Since by (3.12) and (3.13)  $L\tilde{u}_t - h \cdot L\tilde{u} = 0$  in  $Q_\varepsilon$ , then two recent formulas, boundary conditions (3.8), the initial condition (3.18) as well as (3.14) and (3.17) lead to

$$|c_1(x)v_{tt} - \Delta v| \leq M \left[ |\nabla v|(x, t) + |v|(x, t) + \int_0^t (|\nabla v| + |v|)(x, \tau) d\tau \right] \text{ in } \overline{Q}_\varepsilon, \quad (3.19)$$

$$v|_{S_\varepsilon} = \frac{\partial v}{\partial n}|_{S_\varepsilon} = 0, \quad (3.20)$$

$$v(x, 0) = 0, \quad (3.21)$$

where  $M > 0$  is a constant independent on  $v, x, t$ . The idea now is to apply the Carleman estimate of Theorem 2.5 to the problem (3.19)-(3.21) and estimate integrals using Lemma 3.1.

Let the point  $x_0 \in \Omega$ , the number  $P = P(x_0, \Omega)$  be the one defined in (2.61) and  $\eta_0 = \eta_0(d, P, \|\nabla c\|_{C(\overline{\Omega})}) \in (0, 1)$  be the number considered in Theorem 2.5. Choose an arbitrary number  $\eta \in (0, \eta_0)$ . Assuming that  $\varepsilon$  is so small that  $\eta_0\varepsilon^2 < R^2$ , consider the domain  $G_{\eta\varepsilon^2}^+$ ,

$$G_{\eta\varepsilon^2}^+ = \left\{ (x, t) : |x|^2 - \eta t^2 > R^2 - \eta\varepsilon^2, t > 0, |x| < R \right\}.$$

Then  $G_{\eta\varepsilon^2}^+ \subset \overline{Q}_\varepsilon$ . Let  $S_\varepsilon = \partial\Omega \times (0, \varepsilon)$ . The boundary of  $G_{\eta\varepsilon^2}^+$  consists of three parts,

$$\begin{aligned} \partial G_{\eta\varepsilon^2}^+ &= \bigcup_{i=1}^3 \partial_i G_{\eta\varepsilon^2}^+, \\ \partial_1 G_{\eta\varepsilon^2}^+ &= S_\varepsilon \cap \overline{G}_{\eta\varepsilon^2}^+, \\ \partial_2 G_{\eta\varepsilon^2}^+ &= \left\{ |x| \in \left( \sqrt{R^2 - \eta\varepsilon^2}, R \right), t = 0 \right\}, \\ \partial_3 G_{\eta\varepsilon^2}^+ &= \left\{ |x|^2 - \eta t^2 = R^2 - \eta\varepsilon^2, t > 0, |x| < R \right\}. \end{aligned}$$

Square both sides of inequality (3.19), multiply by the function  $\varphi^2(x, t)$  defined in subsection 2.4, apply Theorem 2.5 and Gauss-Ostrogradsky formula. By (2.62) and (3.20) integral over  $\partial_1 G_{\eta\varepsilon^2}^+$  equals zero. By

(2.64) and (3.21) integral over  $\partial_2 G_{\eta\varepsilon^2}^+$  also equals zero. Hence, we obtain with a different constant  $M$

$$\begin{aligned}
C\lambda \int_{G_{\eta\varepsilon^2}^+} (\nabla v)^2 \varphi^2 dxdt + C\lambda^3 \int_{G_{\eta\varepsilon^2}^+} v^2 \varphi^2 dxdt &\leq M \int_{G_{\eta\varepsilon^2}^+} [(\nabla v)^2 + v^2] \varphi^2 dxdt \\
&\leq M \int_{G_{\eta\varepsilon^2}^+} \left( \int_0^t (|\nabla v| + |v|)(x, \tau) d\tau \right)^2 \varphi^2 dxdt \\
&\quad + C\lambda^3 \exp[2\lambda(R^2 - \eta\varepsilon^2)] \int_{\partial_3 G_{\eta\varepsilon^2}^+} [(\nabla v)^2 + v^2] dS_{x,t}, \forall \lambda > \lambda_0.
\end{aligned}
\tag{3.22}$$

For an arbitrary point  $(x, 0) \in \partial_2 G_{\eta\varepsilon^2}^+$  consider the straight line, which is parallel to the  $t$ -axis and passes through the point  $(x, 0)$ . Then this line intersects the hypersurface  $\partial_3 G_{\eta\varepsilon^2}^+$  at the point  $(x, t(x))$ . Hence, applying Lemma 3.1 to any function  $g \in C(\overline{G_{\eta\varepsilon^2}^+})$ , we obtain

$$\int_{G_{\eta\varepsilon^2}^+} \left( \int_0^t |g|(x, \tau) d\tau \right)^2 \varphi^2 dxdt = \int_{\partial_3 G_{\eta\varepsilon^2}^+} dx \left[ \int_0^{t(x)} \left( \int_0^t |g|(x, \tau) d\tau \right)^2 \varphi^2 dt \right] \leq \frac{C}{\lambda} \int_{G_{\eta\varepsilon^2}^+} g^2 \varphi^2 dxdt. \tag{3.23}$$

Using (3.22) and (3.23) and choosing a sufficiently large number  $\lambda_1 = \lambda_1(M, \lambda_0) > \lambda_0$ , we obtain with a different constant  $C$

$$C\lambda \int_{G_{\eta\varepsilon^2}^+} (\nabla v)^2 \varphi^2 dxdt + C\lambda^3 \int_{G_{\eta\varepsilon^2}^+} v^2 \varphi^2 dxdt \leq C\lambda^3 \exp[2\lambda(R^2 - \eta\varepsilon^2)] \int_{\partial_3 G_{\eta\varepsilon^2}^+} [(\nabla v)^2 + v^2] dS_{x,t}, \forall \lambda > \lambda_1.$$

Let  $\delta \in (0, \eta\varepsilon^2)$  be any number. Then  $G_\delta^+ \subset G_{\eta\varepsilon^2}^+$  and  $\varphi^2(x, t) > \exp[2\lambda(R^2 - \delta)]$  in  $G_\delta^+$ . Hence, strengthening the last inequality, we obtain with a different constant  $C$

$$\lambda^3 \exp[2\lambda(R^2 - \delta)] \int_{G_\delta^+} v^2 dxdt \leq C\lambda^3 \exp[2\lambda(R^2 - \eta\varepsilon^2)] \int_{\partial_3 G_{\eta\varepsilon^2}^+} [(\nabla v)^2 + v^2] dS_{x,t}, \forall \lambda > \lambda_1.$$

Dividing by  $\lambda^3 \exp[2\lambda(R^2 - \delta)]$ , we obtain

$$\int_{G_\delta^+} v^2 dxdt \leq C \exp[-2\lambda(\eta\varepsilon^2 - \delta)] \int_{\partial_3 G_{\eta\varepsilon^2}^+} [(\nabla v)^2 + v^2] dS_{x,t}.$$

Setting here  $\lambda \rightarrow \infty$ , we obtain  $v(x, t) = 0$  in  $G_\delta^+$ . Since numbers  $\eta \in (0, \eta_0)$  and  $\delta \in (0, \eta\varepsilon^2)$  are arbitrary ones, then  $v(x, t) = 0$  in  $G_{\eta_0\varepsilon^2}^+$ .

Hence, by (3.16)  $\tilde{u}(x, t) = 0$  in  $G_{\eta_0\varepsilon^2}^+$ . Hence, setting  $t = 0$  in (3.7) and using (3.5), we obtain

$$\tilde{c}(x) = 0 \text{ for } x \in \left\{ |x| \in \left( \sqrt{R^2 - \eta_0\varepsilon^2}, R \right) \right\}. \tag{3.24}$$

Substitute (3.24) in (3.7) and use (??) and (3.8). We obtain for  $(x, t) \in \left\{ |x| \in \left( \sqrt{R^2 - \eta_0 \varepsilon^2}, R \right) \right\} \times (0, T)$

$$L\tilde{u} = c_1(x) \tilde{u}_{tt} - \Delta \tilde{u} - \sum_{j=1}^n a_\alpha(x) D_x^\alpha \tilde{u} = 0, \quad (3.25)$$

$$\tilde{u}(x, 0) = 0, \tilde{u}_t(x, 0) = 0, \quad (3.26)$$

$$\tilde{u}|_{S_T} = \frac{\partial \tilde{u}}{\partial n}|_{S_T} = 0. \quad (3.27)$$

Consider an arbitrary number  $t_0 \in (0, T - \varepsilon)$ . And consider the domain  $G_{\eta_0 \varepsilon^2}(t_0)$ ,

$$G_{\eta_0 \varepsilon^2}(t_0) = \left\{ (x, t) : |x|^2 - \eta_0(t - t_0)^2 > R^2 - \eta_0 \varepsilon^2, t > 0, |x| < R \right\}.$$

Since  $t_0 \in (0, T - \varepsilon)$ , then  $t \in (0, T)$  in this domain. Hence,

$$G_{\eta_0 \varepsilon^2}(t_0) \subset \left\{ |x| \in \left( \sqrt{R^2 - \eta_0 \varepsilon^2}, R \right) \right\} \times (0, T).$$

Hence, using conditions (3.26), (3.27), we can apply Theorems 2.3, 2.5 to the domain  $G_{\eta_0 \varepsilon^2}(t_0)$ . Therefore,  $\tilde{u}(x, t) = 0$  for  $(x, t) \in G_{\eta_0 \varepsilon^2}(t_0)$ . Since  $t_0$  is an arbitrary number of the interval  $(0, T - \varepsilon)$ , then, varying this number, we obtain that

$$\tilde{u}(x, t) = 0 \text{ for } (x, t) \in \left\{ |x| \in \left( \sqrt{R^2 - \eta_0 \varepsilon^2}, R \right) \right\} \times (0, T - \varepsilon).$$

Therefore, we now can replace in (3.7)-(??) sets  $Q_T$  and  $S_T$  with sets  $Q_T^\varepsilon$  and  $S_T^\varepsilon$  respectively, where

$$Q_T^\varepsilon = \left\{ |x| < \sqrt{R^2 - \eta_0 \varepsilon^2} \right\} \times (0, T - \varepsilon), S_T^\varepsilon = \left\{ |x| = \sqrt{R^2 - \eta_0 \varepsilon^2} \right\} \times (0, T - \varepsilon).$$

Next, we repeat the above proof. Hence, we obtain instead of (3.24)

$$\tilde{c}(x) = 0 \text{ for } x \in \left\{ |x| \in \left( \sqrt{R^2 - 2\eta_0 \varepsilon^2}, R \right) \right\}.$$

Since  $\varepsilon > 0$  is sufficiently small, we can choose  $\varepsilon$  such that  $R^2 = k\eta_0 \varepsilon^2$  where  $k = k(R, \varepsilon) \geq 1$  is an integer. Suppose that

$$T > k\varepsilon = \frac{R^2}{\eta_0 \varepsilon}.$$

Then we can repeat this process  $k$  times until the entire domain  $\Omega = \{|x| < R\}$  will be covered. Thus, we obtain that  $\tilde{c}(x) = 0$  in  $\Omega$ . Hence, the right hand side of equation (3.7) is identical zero. This and the standard energy estimate imply that  $\tilde{u}(x, t) = 0$  in  $Q_T$ .  $\square$

An inconvenient point of Theorem 3.1 is that the observation time  $T$  is assumed to be sufficiently large. An experience of the author of working with experimental data [14, 18, 92, 102, 103] indicates that this is not a severe restriction in applications. Indeed, usually the pre-processing procedure of the measured signal leaves only a small portion of the time dependent curve to work with. Still, it is possible to restrict the value of  $T$  via imposing the condition  $f_1(x) \equiv 0$ . The proof of Theorem 3.2 partially uses arguments of works of Imanuvilov and Yamamoto [60, 61].

**Theorem 3.2.** *Assume that all conditions of Theorem 3.1 are satisfied. In addition, assume that the function  $f_1(x) \equiv 0$ . Then Theorem 3.1 holds if*

$$T > \frac{R}{\sqrt{\eta_0}}. \quad (3.28)$$

In particular if  $c(x) \equiv 1$ , then it is sufficient to have  $T > R$ .

**Proof.** Just as in the proof of Theorem 3.1, we consider now only Case 1, keep notations of Theorem 3.1. Introduce the function  $w(x, t) = \tilde{u}_{tt}(x, t)$ . Then by (3.7)-(??)

$$c_1(x) w_{tt} - \Delta w - \sum_{j=1}^n a_\alpha(x) D_x^\alpha w = -\tilde{c}(x) \partial_t^4 u_2, \text{ in } Q_T, \quad (3.29)$$

$$\begin{aligned} w_t(x, 0) &= 0, \\ w|_{S_T} &= \frac{\partial w}{\partial n}|_{S_T} = 0, \\ w(x, 0) &= -\tilde{c}(x) p(x), \end{aligned} \quad (3.30)$$

$$p(x) = c_1^{-1}(x) \left( \Delta f_0(x) + \sum_{|\alpha| \leq 1} a_\alpha(x) D_x^\alpha f_0(x) \right) \neq 0, x \in \bar{\Omega} \quad (3.31)$$

By (3.30) and (3.31)

$$-\tilde{c}(x) = \frac{w(x, 0)}{p(x)} = \frac{1}{p(x)} \left[ w(x, t) - \int_0^t w_t(x, \tau) d\tau \right].$$

Substituting this formula in (3.29) and proceeding similarly with the proof of Theorem 3.1, we obtain that  $\tilde{c}(x) = 0$  in  $\Omega$  and  $w(x, t) = \tilde{u}(x, t) = 0$  in  $Q_T$ .  $\square$

**3.3. MCIPs for parabolic equations.** In this subsection we prove uniqueness theorems for three MCIPs for parabolic PDEs. Unlike the hyperbolic case, conditions imposed on the principal part of the corresponding elliptic operator are not restrictive in two out of three of these problems. The reason is that Carleman estimate can be proven for a quite general parabolic operator, see §1 of Chapter 4 of the book [112]. Below  $C^{k+\beta}, C^{2k+\beta, k+\beta/2}$  are Hölder spaces, where  $k \geq 0$  is an integer and  $\beta \in (0, 1)$ .

**3.3.1. The first MCIP for a parabolic equation.** Denote  $D_T^{n+1} = \mathbb{R}^n \times (0, T)$ . Consider the Cauchy problem for the following parabolic equation

$$c(x) u_t = \Delta u + \sum_{|\alpha| \leq 1} a_\alpha(x) D_x^\alpha u \text{ in } D_T^{n+1}, \quad (3.32)$$

$$u(x, 0) = f_0(x), \quad (3.33)$$

$$c, a_\alpha \in C^\beta(\mathbb{R}^n), \quad c(x) \in [1, d], \quad f_0 \in C^{2+\beta}(\mathbb{R}^n). \quad (3.34)$$

The problem (3.32)-(3.34) has unique solution  $u \in C^{2+\beta, 1+\beta/2}(\bar{D}_T^{n+1})$ , see the book of Ladyzhenskaya, Solonnikov and Uralceva [104]. Just as in Section 3.2, we assume that  $\Omega = \{|x| < R\} \subset \mathbb{R}^n, n \geq 2$ . Let  $\Gamma \subseteq \partial\Omega$  be a part of the boundary of the domain  $\Omega$ ,  $T = \text{const.} > 0$  and  $\Gamma_T = \Gamma \times (0, T)$ .

**The First Parabolic Coefficient Inverse Problem.** Suppose that one of coefficients in equation (3.32) is unknown inside of the domain  $\Omega$  and is known outside of it. Also, assume that all other coefficients in (3.32) are known and conditions (3.33), (3.34) are satisfied. Determine that unknown coefficient inside of  $\Omega$ , assuming that the following functions  $p(x, t)$  and  $q(x, t)$  are known

$$u|_{\Gamma_T} = p(x, t), \quad \frac{\partial u}{\partial n}|_{\Gamma_T} = q(x, t). \quad (3.35)$$

It is yet unclear how to prove a uniqueness theorem for this CIP “straightforwardly”. The reason is that one cannot extend properly the solution of the problem (3.32), (3.33) in  $\{t < 0\}$ . Thus, the idea here is to consider an associated MCIP for the hyperbolic PDE (3.1) using a connection between these two CIPs via

an analog of the Laplace transform. Next, since the Laplace transform is one-to-one, then Theorem 3.1 will provide the desired uniqueness result.

That associated hyperbolic Cauchy problem is

$$c(x) v_{tt} = \Delta v + \sum_{|\alpha| \leq 1} a_\alpha(x) D_x^\alpha v \quad \text{in } D_\infty^{n+1} = \mathbb{R}^n \times (0, \infty), \quad (3.36)$$

$$v|_{t=0} = 0, \quad v_t|_{t=0} = f_0(x). \quad (3.37)$$

In addition to (3.34), we assume that the coefficients  $c(x), a_\alpha(x)$  and the initial condition  $f_0(x)$  are so smooth that the solution  $v$  of the problem (3.36), (3.37) is such that (a)  $v \in C^5(\overline{D}_\infty^{n+1})$  if the function  $c(x)$  is unknown, and (b)  $v \in C^{4+|\alpha|}(\overline{D}_\infty^{n+1})$  if the function  $c(x)$  is known and any of functions  $a_\alpha(x)$  is unknown.

Consider the Laplace-like transform  $\mathcal{R}$  of Reznickaya [132]. Since the publication [132] in 1973 this transform is widely used [14, 86, 112]. The following connection between solutions  $u$  and  $v$  of parabolic and hyperbolic Cauchy problems (3.32), (3.33) and (3.36), (3.37) can be easily verified

$$u(x, t) = \frac{1}{2t\sqrt{\pi t}} \int_0^\infty \exp\left[-\frac{\tau^2}{4t}\right] \tau v(x, \tau) d\tau := \mathcal{R}v. \quad (3.38)$$

Using an analogy with the Laplace transform, one can easily prove that the operator  $\mathcal{R}$  is one-to-one. Hence, given functions  $p, q$  in (3.35), the following two functions  $\overline{p}(x, t)$  and  $\overline{q}(x, t)$  can be uniquely determined

$$v|_{\Gamma_\infty} = \overline{p}(x, t), \quad \frac{\partial v}{\partial n}|_{\Gamma_\infty} = \overline{q}(x, t). \quad (3.39)$$

Therefore, we have reduced the First Parabolic Coefficient Inverse Problem to the hyperbolic CIP (3.36), (3.37), (3.39).

To apply Theorem 3.1, we need to replace  $\Gamma_\infty$  in (3.39) with  $S_\infty$ . To do this, we observe that, using (3.35) and the fact that the unknown coefficient is given outside of the domain  $\Omega$ , one can uniquely determine the function  $u(x, t)$  for  $(x, t) \in (\mathbb{R}^n \setminus \Omega) \times (0, T)$ . Indeed, this follows from Remark 2.2. Therefore we can uniquely determine functions  $u, \partial_n u$  at  $S_T$ . Hence, we can replace in (3.39)  $\Gamma_\infty$  with  $S_\infty$ . Hence, Theorem 3.1 implies Theorem 3.3.

**Theorem 3.3.** *Assume that conditions (3.34) hold. Also, assume that coefficients  $c(x), a_\alpha(x)$  and the initial condition  $f_0(x)$  are so smooth that the solution  $v$  of the problem (3.36), (3.37) is such that:*

*(a)  $v \in C^5(\overline{D}_\infty^{n+1})$  if the function  $c(x)$  is unknown, and (b)  $v \in C^{4+|\alpha|}(\overline{D}_\infty^{n+1})$  if the function  $a_\alpha(x)$  is unknown. Let the domain  $\Omega = \{|x| < R\} \subset \mathbb{R}^n, n \geq 2$ . Denote  $b(x) = 1/\sqrt{c(x)}$ . Let the function  $b(x)$  satisfies conditions (2.59), (2.60). Suppose that conditions of either of Cases 1 or 2 of Theorem 3.1 hold. Then conclusions of Theorem 3.1 are true for the inverse problem (3.32)-(3.35).*

**3.3.2. The second MCIP for the parabolic equation.** Two points of Theorem 3.3 are inconvenient ones. First, one needs to reduce the parabolic CIP to the hyperbolic CIP via inverting the transform (3.38). Second, one needs to use a special form of the elliptic operator in (3.36) with the restrictive condition (2.60) imposed on the coefficient  $c(x)$ . Although (2.60) holds for the case  $c(x) \equiv 1$ , still the question remains whether it is possible to prove uniqueness of an MCIP for the case of a general parabolic operator of the second order. We show in this Section that the latter is possible, provided that one can guarantee the existence of the solution of the parabolic PDE for  $t \in (-T, T)$  and that the function  $u(x, 0)$  is known. On the other hand, it was noticed in the paper of Yamamoto and Zou [150] that the measurement of the temperature at  $t = \theta > 0$  is often easier to achieve than the measurement at the initial time moment  $t = 0$ . Thus, that condition likely has a good applied sense.

Let  $\Omega \subset \mathbb{R}^n$  be either finite or infinite connected domain with the piecewise smooth boundary  $\partial\Omega$ ,  $\Gamma \subseteq \partial\Omega$  be a part of this boundary and  $T = \text{const} > 0$ . Denote  $\Gamma_T^\pm = \Gamma \times (-T, T)$ . Let  $L$  be the elliptic



operator in  $Q_T^\pm$  of the form

$$Lu = \sum_{|\alpha| \leq 2} a_\alpha(x) D_x^\alpha u, (x, t) \in Q_T^\pm, \quad (3.40)$$

$$a_\alpha \in C^1(\overline{\Omega}), |\alpha| = 2, a_\alpha \in C(\overline{\Omega}), |\alpha| = 0, 1, \quad (3.41)$$

$$\mu_1 |\xi|^2 \leq \sum_{|\alpha|=2} a_\alpha(x) \xi^\alpha \leq \mu_2 |\xi|^2, \forall \xi \in \mathbb{R}^n, \forall x \in \overline{\Omega}; \mu_1, \mu_2 = \text{const.} > 0. \quad (3.42)$$

**The Second Parabolic Coefficient Inverse Problem.** Assume that one of coefficients  $a_{\alpha_0}(x)$  of the operator  $L$  is unknown in  $\Omega$ , and all other coefficients of  $L$  are known in  $\Omega$ . Let the function  $u \in C^{4,2}(\overline{Q_T^\pm})$  satisfies the parabolic equation

$$u_t = Lu + F(x, t), \text{ in } Q_T^\pm. \quad (3.43)$$

Determine the coefficient  $a_{\alpha_0}(x)$  for  $x \in \Omega$ , assuming that the function  $F(x, t)$  is known in  $Q_T^\pm$  and that the following functions  $f_0(x)$ ,  $p(x, t)$  and  $q(x, t)$  are known as well

$$u(x, 0) = f_0(x), x \in \Omega, \quad (3.44)$$

$$u|_{\Gamma_T^\pm} = p(x, t), \frac{\partial u}{\partial n}|_{\Gamma_T^\pm} = q(x, t). \quad (3.45)$$

**Theorem 3.4.** Assume that conditions (3.41) and (3.42) are valid and that

$$D_x^{\alpha_0} f_0(x) \neq 0 \text{ in } \overline{\Omega}. \quad (3.46)$$

Then there exists at most one solution  $(u, a_{\alpha_0})$  of the inverse problem (3.43)-(3.45) such that  $u \in C^{4,2}(\overline{Q_T^\pm})$ .

**Proof.** Assume that there exist two solutions  $(u_1, a_{\alpha_0}^{(1)})$  and  $(u_2, a_{\alpha_0}^{(2)})$ . Denote  $\tilde{u} = u_1 - u_2, \tilde{a} = a_{\alpha_0}^{(1)} - a_{\alpha_0}^{(2)}$ . Using (3.43)-(3.45), we obtain

$$\tilde{u}_t - L^{(1)} \tilde{u} = \tilde{a}(x) D^{\alpha_0} u_2, \quad (3.47)$$

$$\tilde{u}(x, 0) = 0, \quad (3.48)$$

$$\tilde{u}|_{\Gamma_T^\pm} = \frac{\partial \tilde{u}}{\partial n}|_{\Gamma_T^\pm} = 0. \quad (3.49)$$

Here  $L^{(1)}$  means that the coefficient  $a_{\alpha_0}(x)$  in the operator  $L$  is replaced with  $a_{\alpha_0}^{(1)}(x)$ . It follows from (3.46) that there exists a small number  $\varepsilon > 0$  such that  $D^{\alpha_0} u_2 \neq 0$  in  $\overline{Q_\varepsilon^\pm}$ . Denote for brevity  $g(x, t) = D^{\alpha_0} u_2(x, t)$ . By (3.47)

$$\tilde{a}(x) = \frac{\tilde{u}_t - L^{(1)} \tilde{u}}{g}, (x, t) \in \overline{Q_\varepsilon^\pm}.$$

Differentiating this equality with respect to  $t$  and denoting

$$h(x, t) = \frac{g_t}{g}, \quad (3.50)$$

we obtain

$$\tilde{u}_{tt} - L^{(1)} \tilde{u}_t = h(\tilde{u}_t - L^{(1)} \tilde{u}). \quad (3.51)$$

We now proceed similarly with the proof of Theorem 3.1. Denote

$$v = \tilde{u}_t - h\tilde{u}. \quad (3.52)$$

Solving the Ordinary Differential Equation (3.52) with the zero initial condition (3.48) and taking into account (3.50), we obtain

$$\tilde{u}(x, t) = \int_0^t K(x, t, \tau) v(x, \tau) d\tau, K(x, t, \tau) = \frac{g(x, t)}{g(x, \tau)}, (x, t) \in \overline{Q}_\varepsilon^\pm \quad (3.53)$$

Next, using (3.49), (3.51) and (3.52), we obtain similarly with (3.19) and (3.20)

$$|v_t - L_0^{(1)}v| \leq M \left[ |\nabla v|(x, t) + |v|(x, t) + \operatorname{sgn}(t) \int_0^t (|\nabla v| + |v|)(x, \tau) d\tau \right] \text{ in } \overline{Q}_\varepsilon^\pm, \quad (3.54)$$

$$v|_{\Gamma_\varepsilon^\pm} = \frac{\partial v}{\partial n}|_{\Gamma_\varepsilon^\pm} = 0, \quad (3.55)$$

where  $L_0^{(1)}$  is the principal part of the operator  $L^{(1)}$ ,  $\operatorname{sgn}(t) = 1$  if  $t > 0$  and  $\operatorname{sgn}(t) = -1$  if  $t < 0$ , and the positive constant  $M$  is independent on  $x, t, v$ . Without loss of generality we assume that

$$\Gamma = \{x_1 = 0, |y| \leq Y\}. \quad (3.56)$$

Otherwise we can still obtain (3.56) for at least a piece of  $\Gamma$  via a change of variables. Consider such a point  $x_0 \in \Omega$  that the straight line which is perpendicular to  $\{x_1 = 0\}$  and passes through  $x_0$ , intersects  $\Gamma$ , and the segment of this straight line which connects  $x_0$  and  $\Gamma$ , lies inside of  $\Omega$ . Without loss of generality we assume that

$$x_0 \in \left\{ x_1 > 0, x_1 + \frac{|y|^2}{2Y^2} < \eta - \alpha \right\}, \quad (3.57)$$

where numbers  $\alpha, \eta \in (0, 1)$ ,  $\alpha < \eta$  were defined in subsection 2.3.

Let  $\varphi(x, t)$  be the function defined in (2.24), (2.25). In (2.24) choose the parameter  $T$  so large that  $\varepsilon^2 / (2T^2) < \eta - \alpha$ . Hence, the domain  $G_\eta \subset Q_\varepsilon^\pm$ , where  $G_\eta$  was defined in subsection 2.3. Square both sides of inequality (3.54), multiply by the function  $\varphi^2(x, t)$ , integrate over the domain  $G_\eta$ . Following Remark 2.2, we use the Carleman estimate of Theorem 2.3, assuming that it is valid for the operator  $L_0^{(1)}$ . In addition, we use (3.55). Also, fix the parameter  $\nu := \nu_0(K, G_\eta, \mu_1, \mu_2)$ . We obtain with a new constant  $M$

$$\begin{aligned} C\lambda \int_{G_\eta} (\nabla v)^2 \varphi^2 dx dt + C\lambda^3 \int_{G_\eta} v^2 \varphi^2 dx dt &\leq M \int_{G_\eta} [(\nabla v)^2 + v^2] \varphi^2 dx dt + M \int_{G_\eta} \left[ \int_0^t (|\nabla v| + |v|) d\tau \right]^2 \varphi^2 dx dt \\ &\quad + C\lambda^3 \exp[2\lambda\eta^{-\nu}] \int_{\partial_2 G_\eta} [(\nabla v)^2 + v^2] dS. \end{aligned}$$

Using Lemma 3.1, we obtain similarly with the proof of inequality (3.23) with a new constant  $M$

$$M \int_{G_\eta} \left[ \int_0^t (|\nabla v| + |v|) d\tau \right]^2 \varphi^2 dx dt \leq \frac{M}{\lambda} \int_{G_\eta} [(\nabla v)^2 + v^2] \varphi^2 dx dt.$$

Hence, choosing sufficiently large  $\lambda_1 = \lambda_1(M, K, G_\eta, \mu_1, \mu_2) > \lambda_0$ , we obtain

$$\lambda^3 \int_{G_\eta} v^2 \varphi^2 dx dt \leq C\lambda^3 \exp[2\lambda\eta^{-\nu}] \int_{\partial_2 G_\eta} [(\nabla v)^2 + v^2] dS, \forall \lambda > \lambda_1. \quad (3.58)$$

Let  $\delta \in (0, \eta - \alpha)$  be so small that the point  $(x_0, 0) \in G_{\eta-\alpha}$ . Then making (3.58) stronger, we obtain

$$\lambda^3 \exp \left[ 2\lambda (\eta - \delta)^{-\nu} \right] \int_{G_{\eta-\delta}} v^2 dx dt \leq C \lambda^3 \exp \left[ 2\lambda \eta^{-\nu} \right] \int_{\partial_2 G_\eta} \left[ (\nabla v)^2 + v^2 \right] dS, \forall \lambda > \lambda_1.$$

Dividing this inequality by  $\lambda^3 \exp \left[ 2\lambda (\eta - \delta)^{-\nu} \right]$ , we obtain

$$\int_{G_{\eta-\delta}} v^2 dx dt \leq C \exp \left\{ -2\lambda \left[ (\eta - \delta)^{-\nu} - \eta^{-\nu} \right] \right\}, \forall \lambda > \lambda_1.$$

Setting here  $\lambda \rightarrow \infty$ , we obtain

$$\int_{G_{\eta-\delta}} v^2 dx dt = 0.$$

Hence,  $v(x, t) = 0$  in  $G_{\eta-\delta}$ . Hence, (3.53) implies that  $\tilde{u}(x, t) = 0$  in  $G_{\eta-\delta}$ . Substituting this in (3.47) and using  $D^{\alpha_0} u_2(x, t) \neq 0$  in  $\overline{G}_{\eta-\delta}$ , we obtain  $\tilde{a}(x) = 0$  in  $G_{\eta-\delta} \cap \{t = 0\}$ . In particular  $\tilde{a}(x_0) = 0$ . It is clear that rotating and moving the coordinate system, one can cover the entire domain  $\Omega$  by paraboloids like  $G_{\eta-\delta} \cap \{t = 0\}$ . Thus,  $\tilde{a}(x) \equiv 0$ . This, (3.47), (3.49), Theorems 2.3, 2.4 and Remark 2.2 imply that  $u_1(x, t) = u_2(x, t)$  in  $Q_T^\pm$ .  $\square$

**3.3.3. An MCIP for a parabolic equation with final over determination.** Let  $D_T^{n+1} = \mathbb{R}^n \times (0, T)$  and  $L$  be the elliptic operator in  $\mathbb{R}^n$ , whose coefficients depend only on  $x$ ,

$$Lu = \sum_{|\alpha| \leq 2} a_\alpha(x) D_x^\alpha u, \quad (3.59)$$

$$a_\alpha \in C^{2+\beta}(\mathbb{R}^n), \beta \in (0, 1), \quad (3.60)$$

$$\mu_1 |\xi|^2 \leq \sum_{|\alpha|=2} a_\alpha(x) \xi^\alpha \leq \mu_2 |\xi|^2, \forall x, \xi \in \mathbb{R}^n, \mu_1, \mu_2 = \text{const.} > 0. \quad (3.61)$$

Consider the following Cauchy problem

$$u_t = Lu \text{ in } D_T^{n+1}, \quad u \in C^{4+\beta, 2+\beta/2}(\overline{D}_T^{n+1}), \quad (3.62)$$

$$u|_{t=0} = f(x) \in C^{4+\beta}(\mathbb{R}^n). \quad (3.63)$$

It is well known that the problem (3.62), (3.63) has unique solution [104].

**The Parabolic Coefficient Inverse Problem with Final Overdetermination.** Let  $T_0 \in (0, T)$  and  $\Omega \subset \mathbb{R}^n$  be a bounded domain. Suppose that the coefficient  $a_{\alpha_0}(x)$  of the operator  $L$  is known inside of  $\Omega$  and is unknown outside of  $\Omega$ . Assume that the initial condition  $f(x)$  is also unknown. Determine both the coefficient  $a_{\alpha_0}(x)$  for  $x \in \mathbb{R}^n \setminus \Omega$  and the initial condition  $f(x)$  for  $x \in \mathbb{R}^n$ , assuming that the following function  $F(x)$  is known

$$F(x) = u(x, T_0), x \in \mathbb{R}^n. \quad (3.64)$$

**Theorem 3.5.** *Assume that conditions (3.59)-(3.61) hold, all coefficients of the operator  $L$  belong to  $C^\infty(\Omega)$ , and*

$$D^{\alpha_0} F(x) \neq 0, \text{ in } \mathbb{R}^n \setminus \Omega.$$

*Then there exists at most one pair vector function  $(u(x, t), a_{\alpha_0}(x), f(x))$  satisfying conditions (3.62)-(3.64).*

**Proof.** Consider the solution of the following hyperbolic Cauchy problem

$$\begin{aligned} v_{tt} &= Lv \text{ in } D_{\infty}^{n+1}, \\ v(x, 0) &= 0, v_t(x, 0) = f(x). \end{aligned}$$

Then by (3.38)  $u = \mathcal{R}v$ . Hence, for any  $x \in \mathbb{R}^n$  the function  $u(x, t)$  is analytic as a function of the real variable  $t > 0$ . We now show that the function  $u(x, t)$  can be uniquely determined for  $(x, t) \in \Omega \times (0, T)$ . Since all coefficients  $a_{\alpha} \in C^{\infty}(\Omega)$ , then  $u \in C^{\infty}(\Omega \times (0, T))$ , see the book of Friedman [54]. Hence, using (3.62) and (3.64), we obtain

$$D_t^{k+1} u(x, T_0) = L^k [F(x)], x \in \Omega, k = 0, 1, \dots$$

This means that one can uniquely determine all  $t$  derivatives of the function  $u(x, t)$  at  $t = T_0$  for all  $x \in \Omega$ . Hence, the analyticity of the function  $u(x, t)$  with respect to  $t$  implies that this function can be uniquely determined for  $(x, t) \in \Omega \times (0, T)$ . Next, applying Theorem 3.4, we obtain that the coefficient  $a_{\alpha_0}(x)$  is uniquely determined in the domain  $\mathbb{R}^n \setminus \Omega$ . The classical theorem about the uniqueness of the solution of the parabolic equation with the reversed time [54, 112] implies that the initial condition  $f(x)$  is also uniquely determined.  $\square$

**3.4. An MCIP for an elliptic equation.** We now consider an elliptic analog of the Second Parabolic Coefficient Inverse Problem. Let  $\Omega \subset \mathbb{R}^n$  be either finite or infinite domain with the piecewise smooth boundary  $\partial\Omega$  and let  $\Gamma \subset \partial\Omega$  be a part of this boundary. Let  $T = \text{const} > 0$ . We keep notations of Section 3.3. Let  $L$  be an elliptic operator in  $\Omega$ ,

$$Lu = \sum_{|\alpha| \leq 2} a_{\alpha}(x) D_x^{\alpha} u, x \in \Omega, \quad (3.65)$$

$$a_{\alpha} \in C^1(\overline{\Omega}), |\alpha| = 2; a_{\alpha} \in C(\overline{\Omega}), |\alpha| = 0, 1, \quad (3.66)$$

$$\mu_1 |\xi|^2 \leq \sum_{|\alpha|=2} a_{\alpha}(x) \xi^{\alpha} \leq \mu_2 |\xi|^2, \forall \xi \in \mathbb{R}^n, \forall x \in \Omega; \mu_1, \mu_2 = \text{const.} > 0. \quad (3.67)$$

**Coefficient Inverse Problem for an Elliptic Equation.** Let the function  $u \in C^2(\overline{Q_T^{\pm}})$  satisfies the following conditions

$$u_{tt} + Lu = F(x, t) \text{ in } Q_T^{\pm}, \quad (3.68)$$

$$u(x, 0) = f_0(x) \text{ in } \Omega, \quad (3.69)$$

$$u|_{\Gamma_T^{\pm}} = p(x, t), \quad \frac{\partial u}{\partial n}|_{\Gamma_T^{\pm}} = q(x, t). \quad (3.70)$$

Assume that the coefficient  $a_{\alpha_0}(x)$  of the operator  $L$  is unknown in  $\Omega$  and all other coefficients are known. Determine the coefficient  $a_{\alpha_0}(x)$  from conditions (3.68)-(3.70).

**Theorem 3.6.** Assume that  $D_x^{\alpha_0} f_0(x) \neq 0$  in  $\overline{\Omega}$  and conditions (3.65)-(3.67) hold. Then there exists at most one pair of functions  $(a_{\alpha_0}(x), u(x, t))$  satisfying (3.68)-(3.70) and such that  $u \in C^{3+|\alpha_0|}(\overline{Q_T^{\pm}})$ .

It follows from Theorem 2.4 and Remark 2.2 that the proof of this theorem is completely similar with the proof Theorem 3.4. Therefore, we omit this proof here.

**4. Published Results About BK.** Given a significant number of publications about BK, it would be quite space consuming to tell details about the topic of each one. Therefore, the author provides short comments about cited papers. An interesting reader is referred to the corresponding paper for detail. Many works cited in this section are devoted to either Lipschitz or Hölder global stability estimates for MCIPs. Hölder stability estimate means  $\|a_1 - a_2\|_X \leq C \|F_1 - F_2\|_Y^{\alpha}, \alpha = \text{const.} \in (0, 1]$ . Here  $a_1$  and  $a_2$  are two unknown coefficients, corresponding to the data  $F_1$  and  $F_2$  respectively and  $X$  and  $Y$  are corresponding Banach spaces. The case  $\alpha = 1$  is called Lipschitz stability estimate, which is obviously stronger than  $\alpha < 1$ .

This is why the Lipschitz stability estimate is usually much harder to prove than the Hölder stability. On the other hand, it was briefly mentioned in earlier papers of the author [76, 82] that usually the Hölder stability follows almost immediately from BK. To do this, one needs to combine BK either with Theorem 2.1 or with similar theorems of Chapter 4 of the book [112]. All stability estimates for finding coefficients mentioned in this section are conditional stability estimates, as it is usually the case in the theory of Ill-Posed problems [5, 14, 49, 70, 145]. In other words, some *a priori* upper bounds are imposed either on certain norms of coefficients of interest or on certain norms of solutions of corresponding PDEs. The constant  $C$  depends on these bounds. In some works the problem of finding an unknown coefficient is replaced with the problem of finding the function  $f(x)$  in the source term  $f(x)P(x, t)$ , where the function  $P(x, t)$  is known. This problem is almost equivalent to a corresponding MCIP, since the interpretation of  $f(x)$  in this case is  $f(x) = a_1(x) - a_2(x)$ , see (3.7) and (3.10). Since the problem of finding the source term is linear, unlike an MCIP, then *a priori* bounds depend on the function  $P(x, t)$ .

**4.1. MCIPs for hyperbolic PDEs.** Lipschitz stability is established for some of these MCIPs. This became possible because the hyperbolic PDE can be solved in both forward and backward directions of time. The idea behind Lipschitz stability estimates is to use various combinations of BK with the Lipschitz stability estimate for the Cauchy problem for the hyperbolic equation with Dirichlet and Neumann data given at the lateral boundary of the time cylinder. For the latter see Theorem 5.1 in Section 5.1 as well as originating papers of Klibanov and Malinsky [81] and Kazemi and Klibanov [72].

For the first time, the Lipschitz stability estimate for an MCIP with single measurement data was obtained by Puel and Yamamoto [131]. The initial boundary value problem in [131] is

$$\begin{aligned} u_{tt} &= \Delta u - p(x)u - f(x)P(x, t), (x, t) \in Q_T, \\ u(x, 0) &= u_t(x, 0) = 0, \\ u|_{S_T} &= 0. \end{aligned} \tag{4.1}$$

The MCIP of [131] consists in finding the source function  $f(x)$ , assuming that the normal derivative  $g(x, t) = \partial_n u|_{S_T}$  is known and  $P(x, 0) \neq 0$  in  $\bar{\Omega}$ . Hence, this MCIP can be obtained from the MCIP of finding the unknown coefficient  $p(x)$  by assuming that  $f(x) = p_1(x) - p_2(x)$ , where  $p_1$  and  $p_2$  are two possible coefficients. In this case in (4.1)  $p = p_2$ ,  $u = u_1 - u_2$ ,  $P = u_1$ , where  $u_1$  and  $u_2$  are solutions of the problem (4.1) with  $p := p_1$  and  $p := p_2$  respectively with the same initial conditions

$$u_1(x, 0) = u_2(x, 0) = P(x, 0), u_{1t}(x, 0) = u_{2t}(x, 0)$$

and the same Dirichlet boundary condition  $u_1|_{S_T} = u_2|_{S_T} = P|_{S_T}$ . It was shown in [131] that if  $\Omega = \{|x| < R\}$  and  $T > 2R$ , then

$$\|f\|_{H^3(\Omega)} \leq C \left( \sum_{j=0}^4 \left\| \partial_t^j g \right\|_{L_2(S_T)}^2 \right)^{1/2}.$$

Isakov and Yamamoto [64] and Imanuvilov and Yamamoto [60, 61] have obtained various Lipschitz stability estimates for MCIPs for hyperbolic PDEs with the principal part of hyperbolic operators  $\partial_t^2 - \Delta$ . Bellassoued [23] has proved Lipschitz stability for an MCIP for the equation  $u_{tt} = c^2(x)Au$ , where  $A$  is a self-adjoint elliptic operator of the second order with  $x$ -dependent coefficients and  $c(x)$  is the unknown coefficient.

Imanuvilov and Yamamoto [62] have considered the case of the determination of the coefficient  $p(x)$  in equation

$$u_{tt} = \operatorname{div}(p(x) \nabla u). \tag{4.2}$$

A new element here, compared with Theorem 3.1, is that both the function  $p$  and its first derivatives are involved in equation (4.2). On the other hand, the machinery of Theorem 3.1 would require to consider derivatives  $p_{x_i}$  as “independent” functions. In turn, this would require to use  $n + 1$  independent initial

conditions. However, only one initial condition was used in [62]. Hölder stability estimate was obtained in [62]. This result was extended by Klivanov and Yamamoto [88] to the Lipschitz stability. The method of [88] is different in some respects from the one of [62], because in [88] a combination of ideas of Theorem 3.1 and Theorem 5.1 was used.

Publications cited above in this section used the assumption that the Dirichlet boundary condition is known on the entire boundary  $S_T$  and the Neumann boundary condition is known on a part of the boundary  $\partial\Omega$  satisfying an appropriate geometrical condition. To obtain logarithmic stability estimates for the case when the Neumann boundary condition is known on an arbitrary piece of the boundary, Bellassoued [23] has proposed to use the so-called “Fourier-Bros-Iagolnitzer integral transformation” (FBI) with respect to  $t$ . FBI transforms the hyperbolic equation in the elliptic one, where the operator  $\partial_t^2 - \Delta$  is replaced with  $\partial_t^2 + \Delta$ . While in [23] this was done for equation  $u_{tt} = \Delta u - p(x)u$ , Bellassoued and Yamamoto [24] have extended this result to the case of equation (4.2). We refer to Robbiano [133, 134] for the introduction of the FBI transformation.

Liu and Triggiani [118] have considered the hyperbolic equation with the damping term  $q(x)u_t$ ,

$$\begin{aligned} u_{tt} &= \Delta u + q(x)u_t, (x, t) \in Q_T, \\ u\left(x, \frac{T}{2}\right) &= u_0(x), u_t\left(x, \frac{T}{2}\right) = u_1(x), \\ \partial_n u \mid_{S_T} &= g(x, t). \end{aligned}$$

The MCIP of [118] consists in determining the coefficient  $q(x)$  from the Dirichlet data  $u|_{\Gamma_T} = f(x, t)$ , where  $\Gamma \subset \partial\Omega$  is a part of the boundary  $\partial\Omega$  satisfying an appropriate geometrical condition. The Lipschitz stability estimate for this MCIP was obtained in [118].

As to other publications about the use of BK for MCIPs for hyperbolic PDEs, we refer to works of Khaïdarov [73], Isakov [65, 66, 67], Doubova and Osses [47], Baudouin, Crépeau and Velein [10], Baudouin, Mercado and Osses [8], Yuan and Yamamoto [152] and Liu and Triggiani [119, 120]. All above results for MCIPs for hyperbolic PDEs are obtained under assumptions like the one in (2.60), which is imposed on the coefficient  $c(x)$  in the principal part of the hyperbolic operator. This is because conditions like (2.60) are the only known ones which guarantee the existence of the Carleman estimate for the hyperbolic case on the one hand, and can be directly analytically verified for generic functions on the other hand. Note that (2.60) is valid of course for the case  $c(x) \equiv 1$ .

**4.2. MCIPs for parabolic PDEs.** A comprehensive survey about the topic of this section as well as about some related topics can be found in the paper of Yamamoto [151]. Thus, the author refers to [151] for further references. It is a long standing well known open problem to prove uniqueness theorems for MCIPs for parabolic PDEs with single measurement in the case when the regular initial condition is given at  $\{t = 0\}$  and the equation is valid only for  $t \in (0, T)$ . This is why the only case when the uniqueness can be currently proven for this type of data is the one of Section 3.3.1, where the inverse Reznickaya’s transform (3.38) was used to obtain the MCIP for a similar hyperbolic PDE. Hence, we discuss in this section only the case when the equation is valid for  $t \in (-T, T)$  and the data are given at  $\{t = 0\}$  as well as on at least a part of the lateral boundary. The only exception is the case of nonlinear parabolic PDEs.

An inconvenience of the conventional Carleman estimate of Theorem 2.4 (Section 2.3) is that it is valid only in the paraboloid  $G_\eta$ , which is a subdomain of the time cylinder  $Q_T^\pm$ . On the other hand, Fursikov and Imanuvilov [55, 58] have proved a radically new Carleman estimate for an arbitrary parabolic operator of the second order. This estimate is valid in the entire time cylinder  $Q_T^\pm$ , although the Carleman Weight Function exponentially decays to zero at  $t \rightarrow \pm T$ . Using this fact, Imanuvilov and Yamamoto [59] have proved, for the first time, the Lipschitz stability estimate for an MCIP for a general parabolic equation with  $x$ -dependent coefficients. In [59] the Dirichlet boundary condition for the solution of the forward problem was known at the entire boundary, whereas the Neumann boundary condition was known on any piece of the boundary. Starting from [59], Carleman estimates of the Fursikov-Imanuvilov type became popular in the inverse problems community, see, e.g. the papers of Baudouin and Puel [7] and Cristofol, Gaitan and Ramoul [45].

Yamamoto [151] has obtained the Hölder stability estimate for the case of the equation  $u_t = \operatorname{div}(p(x) \nabla u)$  with the unknown coefficient  $p(x)$  in the case when two boundary conditions are given at any part  $\Gamma$  of the boundary  $\partial\Omega$ . The main difference between this result and the one of Theorem 3.4 is that the machinery of Theorem 3.4 would require to treat first derivatives of the function  $p(x)$  as independent functions, which would lead, in turn to the necessity to use  $(n+1)$  conditions at  $\{t=0\}$ . Unlike the latter, only one condition at  $\{t=0\}$  is used in [151]. It was pointed out in Remark on page 41 of [151] that if one boundary condition would be known at the entire boundary  $\partial\Omega$  and the second one would be known only at  $\Gamma$ , then the Lipschitz stability estimate would be obtained. In the latter case the Carleman estimate of the Fursikov-Imanuvilov type would be used.

Yamamoto and Zou [150] have considered an MCIP for the equation

$$\begin{aligned} u_t &= \Delta u + p(x) u, (x, t) \in Q_T, \\ u|_{S_T} &= \eta(x, t) \end{aligned}$$

Let  $\omega \subset \Omega$  be subdomain of the domain  $\Omega$ . An interesting new feature of this work is that the inverse problem consists in the simultaneous reconstruction of both the coefficient  $p(x)$  and the initial condition  $\mu(x) = u(x, 0)$ , assuming that the following functions  $f$  and  $g$  are given

$$u(x, \theta) = f(x), u|_{\partial\omega \times (0, T)} = g(x, t),$$

where  $\theta = \text{const.} \in (0, T)$ . First, using the technique of [59], they proved Lipschitz stability estimate for the function  $p(x)$ . Next, they proved logarithmic stability for the initial condition  $\mu(x)$  using the method of logarithmic convexity of Payne [129]. This method works for the case when the corresponding elliptic operator is self-adjoint. In addition, they have constructed a numerical method, which is based on the minimization of the Tikhonov functional. A careful convergence analysis was provided. That analysis was confirmed by a number of numerical experiments.

Surprisingly, Lü [122], has applied BK, for the first time, to an MCIP for the stochastic parabolic PDE.

The assumption of Theorem 3.4 that coefficients of the operator  $L$  are independent on  $t$  was imposed only for the sake of simplicity. In fact, one can allow all coefficients, except of  $a_{\alpha_0}(x)$ , to be dependent on both  $x$  and  $t$ . A direct analog of Theorem 3.4 is valid in this case. To prove it, one should use the Carleman estimate of Theorem 2.5 (for  $L_0^{(1)}$ ), a certain change of variables and the assumption that the target coefficient  $a_{\alpha_0}(x)$  is known for  $x \in \Gamma$ , see works of the author [75, 77, 82] as well as Theorem 1.10.7 in the book [14]. The same is true for Theorem 3.6. This idea was used in the works of the author discussed in the next paragraph.

BK was also applied to MCIPs for nonlinear parabolic PDEs. In the 1d case the author has considered the inverse problem for the equation

$$u_t = F(u_{xx}, u_x, u, q_1(u), \dots, q_n(u)), (x, t) \in (0, 1) \times (0, T),$$

where  $\partial_{u_{xx}} F(y) \neq 0, \forall y \in \mathbb{R}^{n+3}$  [79]. Let  $\{x_i\}_{i=1}^{n+1} \subset (0, 1)$  be a sequence of points, such that  $x_i \neq x_j$  if  $i \neq j$ . The inverse problem in [79] consists in determining the vector function  $q(u) = (q_1, \dots, q_n)(u)$ , assuming that the functions  $f_i(t) = u(x_i, t), i \in [1, n+1]$  are known. It was assumed that  $u_x > 0$  in  $[0, 1] \times [0, T]$ . This inequality can often be established via the maximum principle. Uniqueness theorem was proved in [79]. The first step was to introduce a new spatial variable  $z$  and a new function  $v(z, t)$  via  $u(v(z, t), t) := z$ .

In [85] and Chapter 4 of [86] the author considered an MCIP for the nonlinear parabolic equation

$$u_t = F(u_{x_1 x_1}, \dots, u_{x_n x_n}, u_{x_1}, \dots, u_{x_n}, u, x, t, q(u, x_2, \dots, x_n)), x \in \{x_1 \in (0, 1), y \in \Omega'\}, t \in (0, T),$$

where  $y = (x_2, \dots, x_n)$ , and  $\partial_{u_{x_i x_i}} F(z) \in [\mu_1, \mu_2], \forall z \in \mathbb{R}^{3n+3}$ , where  $\mu_1, \mu_2 = \text{const.} > 0$ . Here  $\Omega' \subseteq \mathbb{R}^{n-1}$  is an arbitrary domain. The following functions  $\varphi_1, \varphi_2, \psi_1, \psi_2$  were given in [85]

$$\varphi_1(y, t) = u(0, y, t), \psi_1(y, t) = u_{x_1}(0, y, t), \varphi_2(y, t) = u(1, y, t), \psi_2(y, t) = u_{x_1}(1, y, t), (y, t) \in \Omega' \times (0, T).$$



It was required to reconstruct the function  $q(u, x_2, \dots, x_n)$ . The first step was again to introduce a new spatial variable  $z$  and a new function  $v(z, y, t)$  via  $u(v(z, y, t), t) := z$ . Next, uniqueness theorem was proved. However, a stability estimate was not established in [85]. Furthermore, unlike the linear case, Hölder stability estimate does not follow automatically from BK in this nonlinear case. Thus, the Hölder stability estimate for a similar inverse problem was proved in the paper of Egger, Engl and Klivanov [48] for the case of the equation

$$u_t = u_{xx} + q(u) + f(x, t)$$

with the unknown function  $q$ . In addition, a numerical reconstruction procedure was developed in [48] via minimizing the Tikhonov functional. Numerical results were also presented in [48]. Other MCIPs for nonlinear parabolic PDEs were treated via BK in Boulakia, Grandmont and Osses [31] and Kaltenbacher and Klivanov [71].

Some other MCIPs for parabolic PDEs were treated by various modifications of BK in papers of Bellassoued and Yamamoto [25], Benabdallah, Dermenjian and Le Rousseau [28], Benabdallah, Gaitan and Le Rousseau [29] and Poisson [130].

Isakov [65, 67] has proved uniqueness of the parabolic MCIP with the final overdetermination without the assumption of Theorem 3.5 of the knowledge of the target coefficient in the domain  $\Omega$ . The initial condition is also known in [65, 67]. BK was not used in [65, 67].

**4.3. MCIPs for the non-stationary Schrödinger equation.** Baudouin and Puel [7] were the first ones who has applied BK to the MCIP for the non-stationary Schrödinger equation

$$\begin{aligned} iu_t + \Delta u + q(x)u &= 0, (x, t) \in Q_T, i^2 = -1, \\ u(x, 0) &= u_0(x), \\ u|_{S_T} &= h(x, t). \end{aligned} \tag{4.3}$$

The inverse problem in [7] consists in determining the coefficient  $q(x)$  from the Neumann boundary condition

$$\partial_n u|_{\Gamma_T} = g(x, t), \Gamma_T = \Gamma \times (0, T), \tag{4.4}$$

where  $\Gamma \subseteq \partial\Omega$  is a part of the boundary satisfying an appropriate geometrical condition. First, following the idea of [59], an analog of the Carleman estimate of Fursikov and Imanuvilov [55, 58] was proved. Next, the Lipschitz stability for the MCIP (4.3), (4.4) was established.

For three other results of the topic of this section we refer to works of Baudouin and Mercado [9], Mercado, Osses and Rosier [123] and Yuan and Yamamoto [153].

**4.4. Non-standard PDEs.** Baudouin, Cerpa, Crépeau and Mercado [11] have considered the Kuramoto-Sivashinsky equation (KS) in 1d with  $Q_T = (0, 1) \times (0, T)$ ,

$$u_t + (\sigma(x)u_{xx})_{xx} + \gamma(x)u_{xx} + u \cdot u_x = g, \tag{4.5}$$

$$u(x, 0) = u_0(x), \tag{4.6}$$

$$u|_{x=0} = h_1(t), u|_{x=1} = h_2(t), \tag{4.7}$$

$$u_x|_{x=0} = h_3(t), u_x|_{x=1} = h_4(t). \tag{4.8}$$

Note that KS is a nonlinear equation. Conditions (4.5)-(4.8) define the forward problem for KS. First, an existence, uniqueness and stability theorem for this problem was proved in [11]. In the inverse problem the following functions were assumed to be known

$$u(x, T_0), u_{xx}(0, t), u_{xxx}(0, t),$$

where  $T_0 = \text{const.} \in (0, T)$ . As to the knowledge of the function  $u(x, T_0)$ , see the arguments in the beginning of Section 4.2. The Lipschitz stability for the inverse problem was proved in [11].

Cavaterra, Lorenzi and Yamamoto [43] considered an MCIP for a PDE whose principal part was a hyperbolic operator  $\partial_t^2 - c^2(x) \Delta$  and lower order terms included Volterra integrals

$$\int_0^t (\cdot) d\tau. \quad (4.9)$$

The Lipschitz stability estimate was obtained in [43]. Romanov and Yamamoto [139] obtained the Hölder stability estimate for an MCIP for a hyperbolic-like PDE with integrals (4.9). We also refer to the work of Buhan and Osses [34], where logarithmic stability estimate for a hyperbolic-like coupler system of PDEs with integrals like the one in (4.9) was obtained, a numerical method, based on the minimization of the Tikhonov functional, was developed, and numerical results were presented.

**4.5. Coupled systems of PDEs.** In this section we do not present coupled systems under discussion because they are well known. Still, we present two systems of PDEs, which are not conventionally known. An MCIP for the Maxwell equations was considered by the author in [78]. Let  $\varepsilon(x)$  and  $\sigma(x)$ ,  $x \in \mathbb{R}^3$  be the dielectric permittivity and the electric conductivity coefficients respectively. It was required to find both of them simultaneously, given the magnetic vector field  $\mathbf{H}(x, t)$  outside of the domain of interest. It was assumed that the magnetic permeability coefficient  $\mu(x) \equiv 1$ . Uniqueness theorem was proved. We also refer to the book of Romanov and Kabanikhin [136] for inverse problems for the Maxwell's system with impulsive sources.

Yamamoto [149] studied an inverse source problem for the Maxwell's system. In this case the unknown source vector function  $\mathbf{f}(x)$  is three dimensional. Both electric and magnetic vector fields were known at the boundary. Uniqueness theorem was proved. We also refer to the papers of Li [113] and Li and Yamamoto [114] for the cases of bi-isotropic and anisotropic Maxwell's system respectively. In [113] the Lipschitz stability estimate was proved, and in the paper [114] the Hölder stability was establishes. Note that in [114] the unknown coefficients are actually matrices of dielectric and magnetic permeability coefficients which are independent on one spatial variable but dependent on time  $t$ .

Bellassoued, Cristofol and Soccorsi [27] obtained Hölder stability estimate for an MCIP for the Maxwell's system. In this case both functions  $\varepsilon(x)$  and  $\mu(x)$  were unknown and  $\sigma(x) \equiv 0$ . Tangential components of both magnetic and electric field were measured at the boundary for two sets of initial conditions, i.e. measurements were conducted twice.

Imanuvilov, Isakov and Yamamoto [63] considered the MCIP of the reconstruction of all three elastic coefficients  $\lambda(x), \mu(x), \rho(x)$  in the time dependent Lamé system. Two sets of initial data were used and they have generated two sets of boundary conditions, which were used as the data for MCIP. The Hölder stability estimate was obtained.

In [63] the data for the inverse problem were given at the entire boundary. Unlike this, Bellassoued, Imanuvilov, and Yamamoto [26] considered the MCIP of recovering of elastic coefficients  $\lambda(x), \mu(x), \rho(x)$  of the time dependent Lamé system in the case when the data for the MCIP are given at an arbitrary part  $\Gamma \subset \partial\Omega$  of the boundary for  $t \in (0, T)$ . In other words, the previous idea of Bellassoued [23] and Bellassoued and Yamamoto [24] (Section 4.1) was extended from a single hyperbolic equation to the case of Lamé system. Similarly with [23, 24] the Fourier-Bros-Iagolnitzer transformation was applied. Logarithmic stability estimate was obtained in [26].

Liu and Triggiani [117] considered the following  $2 \times 2$  coupled system of Schrödinger equations in  $Q_T$

$$\begin{aligned} iw_t + \Delta w &= m(x) \cdot \nabla w + n(x) w + \varsigma(x) \cdot \nabla z + p(x) z, \\ iz_t + \Delta z &= \mu(x) \cdot \nabla z + \sigma(x) z + \psi(x) \cdot \nabla w + q(x) w, \\ w\left(x, \frac{T}{2}\right) &= w_0(x), z\left(x, \frac{T}{2}\right) = z_0(x), \\ \partial_n w|_{S_T} &= g_1(x), \partial_n z|_{S_T} = g_2(x). \end{aligned}$$

The MCIP in [117] consists in the determination of the pair of unknown coefficients  $(p, q)(x)$ ,  $x \in \Omega$  from

the Dirichlet boundary data

$$w|_{\Gamma_T} = f_1(x, t), z|_{\Gamma_T} = f_2(x, t),$$

where  $\Gamma \subset \partial\Omega$  is a part of the boundary satisfying an appropriate geometrical condition. Uniqueness theorem was proved in [117].

Fan, Di Cristo, Jiang and Nakamura [50] proved Lipschitz stability estimate for an MCIP for Navier-Stokes equations. The MCIP consists in recovering of the viscosity function from a single boundary measurement.

Wu and Liu [148] considered the MCIP for the thermoelastic system with memory (also, see their preceding work [147]). That system is

$$\begin{aligned} \mathbf{u}_{tt} - \alpha \Delta \mathbf{u} - \beta \nabla \operatorname{div} \mathbf{u} + \gamma \nabla v &= \sigma(x, t) \mathbf{p}(x), (x, t) \in Q_T, \\ v_t - \int_0^t k(t - \tau) \Delta v(x, \tau) d\tau + \gamma \operatorname{div} \mathbf{u}_t &= 0, (x, t) \in Q_T, \\ \mathbf{u}(x, 0) &= \mathbf{u}_0(x), \mathbf{u}_t(x, 0) = \mathbf{u}_1(x), v(x, 0) = v_1(x), \\ \mathbf{u}|_{S_T} &= 0, v|_{S_T} = 0. \end{aligned}$$

Here  $\mathbf{u} = (u_1, u_2, u_3)^T$  and  $v$  are displacement and temperature respectively. Let  $\omega \subset \Omega$  be a subdomain. The MCIP in [148] consists in the recovery of the source term  $\mathbf{p}(x)$ , given the vector function  $\mathbf{f}(x, t) = \mathbf{u}(x, t)$  for  $(x, t) \in \omega \times (0, T)$ . The function  $\sigma(x, t)$  is assumed to be known. Lipschitz stability estimate for this MCIP was obtained in [148].

Cristofol, Gaitan and Ramoul [45] considered the following parabolic  $2 \times 2$  system

$$\begin{aligned} u_t &= \Delta u + a(x)u + b(x)v, (x, t) \in Q_T, \\ v_t &= \Delta v + c(x)u + d(x)v, (x, t) \in Q_T, \\ u(x, 0) &= u_0(x), v(x, 0) = v_0(x), \\ u|_{S_T} &= g(x, t), v|_{S_T} = h(x, t). \end{aligned}$$

This is a reaction-diffusion system. Let  $\omega \subset \Omega$  be a subdomain,  $t_0 \in (0, T)$  and  $T' = (t_0 + T)/2$ . The data for the MCIP in [45] are the following functions

$$\Delta u(x, T'), u(x, T'), v(x, T'), v_t|_{\omega \times (t_0, T)}.$$

The MCIP consists in determining the vector function  $(b, u_0, v_0)$ . The function  $b$  can be replaced with the function  $a$ . A new point of [45] is that only one function  $v_t$  is measured in  $\omega$ , unlike the conventional way of measuring both functions  $u$  and  $v$ . The Lipschitz stability estimate for the coefficient  $b(x)$  as well as the logarithmic stability estimate for initial conditions  $u_0(x), v_0(x)$  were obtained in [45].

**5. Stability Estimates for Hyperbolic Equations and Inequalities With Lateral Cauchy Data and Thermoacoustic Tomography.** It was shown in Section 2.2 how Carleman estimates lead to Hölder stability estimates for ill-posed Cauchy problems for PDEs, including inequalities. In this section we obtain Lipschitz and logarithmic stability estimates for the case of hyperbolic PDEs. We also show how these estimates help to specify QRM for the hyperbolic case. In Section 5.5 we cite related published results.

**5.1. Lipschitz stability.** As usual, we consider the simplest case when the domain of interest  $\Omega$  is a ball,  $\Omega = \{|x| < R\} \subset \mathbb{R}^n$ , where  $R = \text{const.} > 0$ . Let functions  $c(x), b_j(x, t)$  satisfy the following conditions

$$c(x) \in C^1(\overline{\Omega}), c(x) \neq 0 \text{ in } \overline{\Omega}, \quad (5.1)$$

$$b_j \in C(\overline{Q_T^\pm}), j = 0, \dots, n+1. \quad (5.2)$$

Let the function  $u \in C^2(\overline{Q_T^\pm})$  and the function  $f \in C(\overline{Q_T^\pm})$ . Suppose that the function  $u$  is a solution of the following hyperbolic equation

$$u_{tt} = c^2(x) \Delta u + \sum_{j=1}^{n+1} b_j(x, t) u_{x_j} + b_0(x, t) u + f(x, t), (x, t) \in Q_T^\pm, \quad (5.3)$$

where  $u_{n+1} := u_t$ . Consider Dirichlet and Neumann boundary conditions for the function  $u$  at the lateral side  $S_T^\pm$  of the time cylinder  $Q_T^\pm$ ,

$$u|_{S_T^\pm} = p(x, t), \partial_n u|_{S_T^\pm} = q(x, t). \quad (5.4)$$

**Problem 5.1.** Given conditions (5.3), (5.4), estimate the function  $u \in C^2(\overline{Q_T^\pm})$  in the time cylinder  $Q_T^\pm$  via functions  $p, q$  and  $f$ .

The of this section works for a more general case of a hyperbolic inequality. Specifically, we consider the following problem.

**Problem 5.2.** Let  $A = \text{const.} > 0$ . Let the function  $u \in C^2(\overline{Q_T^\pm})$  satisfies the following pointwise inequality in the cylinder  $Q_T^\pm$

$$|u_{tt} - c^2(x) \Delta u| \leq A(|\nabla u| + |u_t| + |u| + |f|), \forall (x, t) \in Q_T^\pm, \quad (5.5)$$

where the function  $f \in L_2(Q_T^\pm)$ . Estimate the function  $u$  via functions  $p, q$  and  $f$ .

Since Problem 5.2 is more general than Problem 5.1, we study only Problem 5.2 in this section. Theorem 4.1 provides the Lipschitz stability estimate for Problem 5.2. We now reformulate condition (2.60) for the case of the operator  $\partial_t^2 - c^2(x) \Delta$  in a stronger form as

$$(x, \nabla c^{-2}(x)) \geq \alpha = \text{const.} > 0 \text{ in } \overline{\Omega}, \quad (5.6)$$

where  $\alpha > 0$  is a certain number and  $(\cdot, \cdot)$  is the scalar product in  $\mathbb{R}^n$ . Hence, there exists a sufficiently small number  $\varepsilon = \varepsilon(\alpha, \|\nabla c^{-2}\|_{C^1(\overline{\Omega})}) \in (0, R)$  such that

$$(x - x_0, \nabla c^{-2}(x)) \geq \frac{\alpha}{2} > 0 \text{ in } \overline{\Omega}, \forall x_0 \in \{|x_0| \leq \varepsilon\}. \quad (5.7)$$

Inequality (5.7) guarantees the Carleman estimate for the operator  $\partial_t^2 - c^2(x) \Delta$  (Theorem 2.5). This estimate is also guaranteed if  $c \equiv \text{const.} \neq 0$ .

**Theorem 5.1.** Let the domain  $\Omega = \{|x| < R\}$ . Let  $\alpha > 0$  and  $d > 1$  be certain numbers. Let

$$c \in C^1(\overline{\Omega}), c^{-2}(x) \in [1, d], \|\nabla c^{-2}\|_{C^1(\overline{\Omega})} \leq d. \quad (5.8)$$

In the case  $c \neq \text{const.}$  we assume that condition (5.6) is fulfilled. Let the function  $u \in C^2(\overline{Q_T^\pm})$  satisfies inequality (5.5) with Dirichlet and Neumann boundary conditions (5.4). Then there exists a constant  $\eta_0 = \eta_0(R, d, \alpha) \in (0, 1]$  such that if

$$T > \frac{R}{\sqrt{\eta_0}}, \quad (5.9)$$

then with a constant  $K = K(A, R, T, d, \alpha) = \text{const.} > 0$  the following Lipschitz stability estimate holds for the function  $u$ ,

$$\|u\|_{H^1(Q_T^\pm)} \leq K \left[ \|p\|_{H^1(S_T^\pm)} + \|q\|_{L_2(S_T^\pm)} + \|f\|_{L_2(Q_T^\pm)} \right]. \quad (5.10)$$

In particular, if  $c(x) \equiv 1$ , then one can take  $\eta_0 = 1$  and in (5.9)  $T > R$ .

**Proof.** In this proof  $K = K(A, R, T, d, \alpha)$  denotes different positive constants depending on listed parameters. We note first that the constant  $P(x_0, \Omega)$  in (2.61) can be estimated as  $P(x_0, \Omega) \leq 2R$ . Hence, we can set in Theorem 2.5  $\eta_0 = \eta_0(R, d, \|c^{-2}\|_{C^1(\overline{\Omega})}) \in (0, 1)$ . By (5.9) we can choose a sufficiently small  $\varepsilon$  in (5.7) and then choose  $\eta$  such that

$$\eta \in \left( \frac{(R + \varepsilon)^2}{T^2}, \eta_0 \right) \subset (0, 1), \quad (5.11)$$

Similarly with Section 2.4 let  $\gamma = \text{const} \in (0, \varepsilon^2/9)$ ,

$$\begin{aligned} \xi(x, t) &= |x|^2 - \eta t^2, \varphi(x, t) = \exp[\lambda \xi(x, t)], (x, t) \in Q_T^\pm, \\ G_\gamma &= \{\xi(x, t) > \gamma, x \in \Omega\} = \{|x|^2 - \eta t^2 > \gamma, x \in \Omega\}, \end{aligned}$$

where  $\lambda > 1$  is a large parameter which we define later. By (5.11)

$$\overline{G}_\gamma \subset \{|t| < T\}. \quad (5.12)$$

Choose a sufficiently small number  $\delta$  such that  $\gamma + 3\delta \in (0, \varepsilon^2/9)$ . Hence,

$$G_{\gamma+3\delta} \neq \emptyset \text{ and } G_{\gamma+3\delta} \subset G_{\gamma+2\delta} \subset G_{\gamma+\delta} \subset G_\gamma. \quad (5.13)$$

Introduce a function  $\chi_\delta \in C^2(\overline{Q}_T^\pm)$  satisfying

$$\chi_\delta(x, t) = \begin{cases} 1, & (x, t) \in G_{\gamma+2\delta}, \\ 0, & (x, t) \in Q_T^\pm \setminus G_{\gamma+\delta}, \\ \text{between 0 and 1} & \text{otherwise.} \end{cases} \quad (5.14)$$

The existence of such functions is well known from the Real Analysis course. Let

$$v(x, t) = u(x, t) \chi_\delta(x, t). \quad (5.15)$$

Multiplying both sides of (5.5) by  $\chi_\delta$  and using (5.4), (5.12), (5.14) and (5.15), we obtain

$$|v_{tt} - c^2(x) \Delta v| \leq K(|\nabla v| + |v_t| + |v| + |f|) + K(1 - \chi_\delta)(|\nabla u| + |u_t| + |u|), (x, t) \in G_\gamma, \quad (5.16)$$

$$v|_{S_T^\pm} = \chi_\delta \varphi, \partial_n v|_{S_T^\pm} = \chi_\delta \psi + \varphi \partial_n \chi_\delta. \quad (5.17)$$

Squaring both sides of (5.16) and using Theorem 2.5, we obtain

$$\begin{aligned} & K(|\nabla v|^2 + v_t^2 + v^2 + f^2) \varphi^2 + K(1 - \chi_\delta)(|\nabla u|^2 + u_t^2 + u^2) \varphi^2 \\ & \geq C\lambda(|\nabla v|^2 + v_t^2) \varphi^2 + C\lambda^3 v^2 \varphi^2 + \text{div } U + V_t, \text{ in } G_\gamma, \forall \lambda \geq \lambda_0. \end{aligned}$$

where the vector function  $(U, V)$  satisfies conditions (2.62), (2.63) with the replacement of  $u$  by  $v$ . Hence, (5.14) implies that  $U = V = 0$  on  $\{(x, t) : \xi(x, t) = \gamma, x \in \Omega\}$ . Hence, integrating the latter inequality over  $G_\gamma$  and using Gauss-Ostrogradsky formula and (5.17), we obtain

$$C \int_{G_\gamma} \lambda(|\nabla v|^2 + v_t^2) \varphi^2 dx dt + C\lambda^3 \int_{G_\gamma} v^2 \varphi^2 dx dt \leq K \int_{G_\gamma} (|\nabla v|^2 + v_t^2 + v^2 + g^2) \varphi^2$$

$$+K \exp [2\lambda (\gamma + 2\delta)] \|u\|_{H^1(Q_T^\pm)}^2 + K e^{2\lambda R^2} \left( \|p\|_{H^1(S_T^\pm)}^2 + \|q\|_{L_2(S_T^\pm)}^2 \right).$$

Let  $\lambda_0 > 1$  be the number of Theorem 2.5. There exists a number  $\lambda_1 = \lambda_1(\lambda_0, C, K) \geq \lambda_0$  such that  $C\lambda/2 > K, \forall \lambda \geq \lambda_1$ . Hence,

$$\lambda \int_{G_\gamma} \left( |\nabla v|^2 + v_t^2 + v^2 \right) \varphi^2 dxdt \leq$$

$$\leq K \exp [2\lambda (\gamma + 2\delta)] \|u\|_{H^1(Q_T^\pm)}^2 + K e^{2\lambda R^2} \left[ \|p\|_{H^1(S_T^\pm)}^2 + \|q\|_{L_2(S_T^\pm)}^2 + \|g\|_{L_2(S_T^\pm)}^2 \right].$$

By (5.13) and (5.14)

$$\begin{aligned} \lambda \int_{G_\gamma} \left( |\nabla v|^2 + v_t^2 + v^2 \right) \varphi^2 dxdt &\geq \lambda \int_{G_{\gamma+3\delta}} \left( |\nabla v|^2 + v_t^2 + v^2 \right) \varphi^2 dxdt \\ &\geq \exp [2\lambda (\gamma + 3\delta)] \|u\|_{H^1(G_{\gamma+3\delta})}^2. \end{aligned}$$

Hence,

$$\|u\|_{H^1(G_{\gamma+2\delta})}^2 \leq K \exp (-2\lambda\delta) \|u\|_{H^1(Q_T^\pm)}^2 + K e^{2\lambda R^2} \left[ \|p\|_{H^1(S_T^\pm)}^2 + \|q\|_{L_2(S_T^\pm)}^2 + \|f\|_{L_2(Q_T^\pm)}^2 \right]. \quad (5.18)$$

Note that

$$G_{\gamma+3\delta} \cap \{t = 0\} = \left\{ x : |x| \in \left( \sqrt{\gamma + 3\delta}, R \right) \right\} \supset \left\{ x : |x| \in \left( \frac{\varepsilon}{3}, R \right) \right\}. \quad (5.19)$$

Choose now a point  $x_0$  such that  $|x_0| = 3\sqrt{\gamma + 3\delta}$ . Then  $|x_0| < \varepsilon$ . Consider now an arbitrary point  $y \in \{|x| \leq \sqrt{\gamma + 3\delta}\}$ . Then

$$|y - x_0| \geq |x_0| - |y| = 3\sqrt{\gamma + 3\delta} - |y| \geq 3\sqrt{\gamma + 3\delta} - \sqrt{\gamma + 3\delta} = 2\sqrt{\gamma + 3\delta} > \sqrt{\gamma + 3\delta}.$$

Hence,

$$\left\{ |x| \leq \sqrt{\gamma + 3\delta} \right\} \subset \left\{ |y - x_0| > \sqrt{\gamma + 3\delta} \right\}. \quad (5.20)$$

Introduce now the domain  $G_\gamma(x_0)$  as

$$G_\gamma(x_0) = \left\{ |x - x_0|^2 - \eta t^2 > \gamma, x \in \Omega \right\}.$$

It follows from (5.19) and (5.20) that there exists a sufficiently small number  $\sigma = \sigma(\varepsilon)$  such that

$$\{t \in (0, \sigma)\} \subset [G_{\gamma+3\delta} \cup G_{\gamma+3\delta}(x_0)]. \quad (5.21)$$

Since  $\gamma + 3\delta \in (0, \varepsilon^2/9)$  and  $|x_0| = 3\sqrt{\gamma + 3\delta}$ , then (5.11) implies that  $\overline{G}_\gamma(x_0) \subset \{|t| < T\}$ . Next, we use (5.7) and Theorem 2.5 and obtain similarly with (5.18)

$$\|u\|_{H^1(G_{\gamma+3\delta}(x_0))}^2 \leq K \exp (-2\lambda\delta) \|u\|_{H^1(Q_T^\pm)}^2 + K e^{2\lambda R^2} \left[ \|p\|_{H^1(S_T^\pm)}^2 + \|q\|_{L_2(S_T^\pm)}^2 + \|f\|_{L_2(Q_T^\pm)}^2 \right].$$

Combining this with (5.18), we obtain

$$\|u\|_{H^1(G_{\gamma+2\delta} \cup G_{\gamma+2\delta}(x_0))}^2 \leq K \exp (-2\lambda\delta) \|u\|_{H^1(Q_T^\pm)}^2 + K e^{2\lambda R^2} \left[ \|p\|_{H^1(S_T^\pm)}^2 + \|q\|_{L_2(S_T^\pm)}^2 + \|f\|_{L_2(Q_T^\pm)}^2 \right].$$

This, (5.21) and the mean value theorem imply that there exists a number  $t_0 \in [0, \sigma]$  such that

$$\begin{aligned} & \|u(x, t_0)\|_{H^1(\Omega)}^2 + \|u_t(x, t_0)\|_{L_2(\Omega)}^2 \\ & \leq K \exp(-2\lambda\delta) \|u\|_{H^1(Q_T^\pm)}^2 + K e^{2\lambda R^2} \left[ \|p\|_{H^1(S_T^\pm)}^2 + \|q\|_{L_2(S_T^\pm)}^2 + \|f\|_{L_2(Q_T^\pm)}^2 \right]. \end{aligned} \quad (5.22)$$

Let  $u_{tt} - c^2(x)u := Z(x, t)$ . Then by (5.5)

$$|Z| \leq A(|\nabla u| + |u_t| + |u| + |f|), \forall (x, t) \in Q_T^\pm. \quad (5.23)$$

Consider the initial boundary value problem with reversed time

$$u_{tt} - c^2(x)u = Z(x, t), \forall (x, t) \in \{x \in \Omega, t \in (-T, t_0)\}, \quad (5.24)$$

$$u(x, t_0) = u_0(x), u_t(x, t_0) = u_1(x), \quad (5.25)$$

$$u|_{(x,t) \in \partial\Omega \times (-T, t_0)} = p(x, t). \quad (5.26)$$

Next, consider the same initial boundary value problem but in the time cylinder  $(x, t) \in \{x \in \Omega, t \in (t_0, T)\}$ . Recall that the hyperbolic equation can be solved in both positive and negative directions of time. Hence, the standard method of energy estimates being applied to two latter problems combined with inequalities (5.22) and (5.23) leads to

$$\|u\|_{H^1(Q_T^\pm)}^2 \leq K \exp(-2\lambda\delta) \|u\|_{H^1(Q_T^\pm)}^2 + K e^{2\lambda R^2} \left[ \|p\|_{H^1(S_T^\pm)}^2 + \|q\|_{L_2(S_T^\pm)}^2 + \|f\|_{L_2(Q_T^\pm)}^2 \right]. \quad (5.27)$$

Choosing  $\lambda = \lambda(K, \delta)$  so large that  $K \exp(-2\lambda\delta) \leq 1/2$ , we obtain the target estimate (5.10) from (5.27).  $\square$

To apply QRM (Section 5.4.2), we need Theorem 5.2. The proof of this theorem is almost identical with the proof of Theorem 5.1.

**Theorem 5.2.** *Let the function  $u \in C^2(\overline{Q_T^\pm})$  satisfies the Dirichlet and Neumann boundary conditions (5.4) as well as the following integral inequality*

$$\int_{Q_T^\pm} \left( u_{tt} - c^2(x) \Delta u - \sum_{j=1}^{n+1} b_j(x, t) u_{x_j} - b_0(x, t) u \right)^2 dx dt \leq S^2$$

*and the rest of conditions of Theorem 5.1 is in place. Here  $S = \text{const.} > 0$ . Then there exists a constant  $\eta_0 = \eta_0(R, d, \alpha) \in (0, 1]$  such that if  $T > R/\sqrt{\eta_0}$ , then with a constant  $K = K(A, R, T, d, \alpha) = \text{const.} > 0$  the following Lipschitz stability estimate holds for the function  $u$ ,*

$$\|u\|_{H^1(Q_T^\pm)} \leq B \left[ \|p\|_{H^1(S_T^\pm)} + \|q\|_{L_2(S_T^\pm)} + S \right].$$

*In particular, if  $c(x) \equiv 1$ , then one can take  $\eta_0 = 1$  and  $T > R$ .*

**5.2. Thermoacoustic tomography.** In thermoacoustic tomography (TAT) a short radio frequency pulse is sent in a biological tissue, see papers of Agranovsky and Kuchment [2] and Finch and Rakesh [53]. Some energy is absorbed. It is well known that malignant legions absorb much more energy than healthy ones. Then the tissue expands and radiates a pressure wave. The propagation of this wave can be modeled as the solution of the following Cauchy problem

$$u_{tt} = c^2(x) \Delta u, x \in \mathbb{R}^3, \quad (5.28)$$

$$u(x, 0) = f(x), u_t(x, 0) = 0. \quad (5.29)$$

The function  $u(x, t)$  is measured by transducers at certain locations either at the boundary of the medium of interest or outside of this medium. The function  $f(x)$  characterizes the absorption of the medium. Hence, if



one would know the function  $f(x)$ , then one would know locations of malignant spots. The inverse problem consists in determining the initial condition  $f(x)$  using those measurements. Hence, we obtain the following inverse problem

**Inverse Problem 5.1 (IP5.1).** Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain with  $\partial\Omega \in C^1$ . Consider the Cauchy problem (5.28), (5.29). Suppose that

$$f(x) = 0, x \in \mathbb{R}^3 \setminus \Omega. \quad (5.30)$$

Determine the function  $f(x)$  inside of the domain  $\Omega$  assuming that the following function  $p(x, t)$  is known

$$u|_{S_T} = p(x, t). \quad (5.31)$$

We show below in this section how Theorem 5.1 implies the Lipschitz stability estimate for IP5.1. Assume that  $c(x) = 1$  for  $x \in \mathbb{R}^3 \setminus \Omega$ . Then (5.28)-(5.31) imply that the function  $u(x, t)$  is the solution of the following initial boundary value problem in  $(\mathbb{R}^3 \setminus \Omega) \times (0, T)$

$$\begin{aligned} u_{tt} &= \Delta u, x \in \mathbb{R}^3 \setminus \Omega, t \in (0, T), \\ u(x, 0) &= u_t(x, 0) = 0, x \in \mathbb{R}^3 \setminus \Omega, \\ u|_{S_T} &= p(x, t). \end{aligned} \quad (5.32)$$

Under certain well known conditions the solution  $u \in H^2((\mathbb{R}^3 \setminus \Omega) \times (0, T))$  of this problem exists, is unique and depends continuously on the boundary data  $p(x, t)$ . Just as above consider the case  $\Omega = \{|x| < R\}$ . For  $|x| \geq R$  let  $\bar{p}(x, t) = p(x, t)|x|R^{-1}\rho(x)$ , where the function  $\rho \in C^2(|x| \geq R)$ ,  $\rho(x) = 0$  for  $x \in \{|x| \geq 3R\}$  and  $\rho(x) = 1$  for  $x \in \{|x| \in [R, 2R]\}$ . Let  $v(x, t) = u(x, t) - \bar{p}(x, t)$ . Substituting  $v$  in equations (5.32) and using the standard method of energy estimates (see, e.g. the book of Ladyzhenskaya [106] for this method), we obtain the following stability estimate  $\|u\|_{H^2((\mathbb{R}^3 \setminus \Omega) \times (0, T))} \leq C \|p\|_{H^3(S_T)}$ . Let  $q(x, t) = \partial_n u|_{S_T}$ . Hence, trace theorem implies that

$$\|q\|_{L_2(S_T)} = \|\partial_n u\|_{L_2(S_T)} \leq C \|p\|_{H^3(S_T)}. \quad (5.33)$$

Here  $C > 0$  denotes different positive constants depending only on  $R$  and the function  $\rho(x)$ .

Consider now the even extension  $\tilde{u}(x, t)$  with respect to  $t$  of the function  $u(x, t)$ . Assuming that  $u \in C^2(\bar{Q}_T)$ , we obtain  $\tilde{u} \in C^2(\bar{Q}_T^\pm)$  and also

$$\tilde{u}_{tt} = c^2(x) \Delta \tilde{u}, (x, t) \in Q_T^\pm, \quad (5.34)$$

$$\tilde{u}|_{S_T^\pm} = \tilde{p}(x, t), \partial_n \tilde{u}|_{S_T^\pm} = \tilde{q}(x, t), \quad (5.35)$$

where functions  $\tilde{p}$  and  $\tilde{q}$  are even extensions of functions  $p$  and  $q$  respectively. In addition, by (5.29)

$$\tilde{u}(x, 0) = f(x). \quad (5.36)$$

Hence, Theorem 5.1, (5.33)-(5.36) and trace theorem imply Theorem 5.3. We need here  $u \in H^4(\mathbb{R}^3 \times (0, T))$  in order to make sure that the function  $p \in H^3(S_T)$ .

**Theorem 5.3.** *Let the domain  $\Omega = \{|x| < R\}$ . Let the function  $c \in C^1(\mathbb{R}^3)$ ,  $c = 1$  in  $\mathbb{R}^3 \setminus \Omega$  and also  $c$  satisfies conditions (5.8). In the case  $c \neq \text{const.}$  we assume that condition (5.6) is fulfilled. Suppose that functions  $c(x)$  and  $f(x)$  are such that there exists the solution  $u \in C^2(\mathbb{R}^3 \times [0, T]) \cap H^4(\mathbb{R}^3 \times (0, T))$  of the Cauchy problem (5.28), (5.29). Also, let condition (5.30) be satisfied. Then there exists a constant  $\eta_0 = \eta_0(R, d, \alpha) \in (0, 1]$  such that if  $T > R/\sqrt{\eta_0}$ , then with a constant  $K = K(A, R, T, d, \alpha) > 0$  the following Lipschitz stability estimate holds for IP5.1*

$$\|f\|_{L_2(\Omega)} \leq K \|p\|_{H^3(S_T)}.$$

In particular, if  $c(x) \equiv 1$ , then one can take  $\eta_0 = 1$  and  $T > R$ .

**5.3. Logarithmic stability in the case of a general hyperbolic operator of the second order with  $x$ -dependent coefficients.** Condition (5.6) is used in Theorem 5.1 because it is linked with the existence of the Carleman estimate for the hyperbolic case, see the end of Section 4.1. Clearly (5.6) is a restrictive condition. Therefore, the next question is whether a stability estimate can be obtained for an analog of IP5.1 in the case of an arbitrary hyperbolic operator of the second order. For the first time, this question was positively addressed by the author in [95]. We formulate main results of [95] in this section without proofs.

The idea is to apply an analog of the above Reznickaya's transform (3.38). This way the hyperbolic PDE is transformed in a similar parabolic PDE. And the function  $f(x)$  becomes the initial condition for that parabolic PDE. On the other hand, logarithmic stability estimates for the inverse problem of determination of the initial condition of a general parabolic equation from lateral Cauchy data were obtained by Klivanov [89] in the case of a finite domain  $\Omega \subset \mathbb{R}^n$  and by Klivanov and Tikhonravov [?] in the case of an infinite domain  $\Omega \subseteq \mathbb{R}^n$ . Thus, modifications of these results can be applied. Results of both publications [89, ?] were obtained via Carleman estimates. The difference between logarithmic stability estimates for initial conditions in [89, ?] and those in the book of Payne [129] is that in [89, ?] the case of a general elliptic operator with coefficients depending on  $(x, t)$  was considered. On the other hand, the technique of [129] works only for self-adjoint elliptic operators with coefficients depending only on  $x$ . Since the method of [89, ?] works not only for parabolic PDEs but for integral inequalities as well (see Theorem 2.2 in [95] as well as Theorem 2.3 in Section 2.2 above), then it enables us to prove convergence of the QRM, unlike the technique of [129].

We refer to Li, Yamamoto and Zou [115] for another logarithmic stability estimate of the initial condition of a parabolic equation with the self-adjoint operator  $L$  in a finite domain. A Carleman estimate was also used in this reference. An interesting feature of [115] is that observations are performed on an internal subdomain for times  $t \in (\tau, T)$  where  $\tau > 0$ . In addition, a numerical method was developed in [115].

**5.3.1. Statements of inverse problems.** Let  $\Omega \subset \{x_1 > 0\}$  be a bounded domain with the boundary  $\partial\Omega \in C^3$ . Denote  $P = \{x_1 = 0\}$ ,  $P_T = P \times (0, T)$ ,  $\forall T > 0$ . Let  $k \geq 0$  be an integer and  $\alpha \in (0, 1)$ . Consider the elliptic operator  $L$  of the second order,

$$Lu = \sum_{i,j=1}^n a_{i,j}(x) u_{x_i x_j} + \sum_{j=1}^n b_j(x) u_{x_j} + b_0(x) u, x \in \mathbb{R}^n, \quad (5.37)$$

$$a_{i,j} \in C^{k+\alpha}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n), b_j, b_0 \in C^{k+\alpha}(\mathbb{R}^n), k \geq 2, \alpha \in (0, 1), \quad (5.38)$$

$$\mu_1 |\eta|^2 \leq \sum_{i,j=1}^n a_{i,j}(x) \eta_i \eta_j \leq \mu_2 |\eta|^2, \forall x, \eta \in \mathbb{R}^n; \mu_1, \mu_2 = \text{const.} > 0. \quad (5.39)$$

Let the function  $f(x)$  be such that

$$f \in C^p(\mathbb{R}^n), p \geq 3, f(x) = 0, x \in \mathbb{R}^n \setminus \Omega. \quad (5.40)$$

Consider the following Cauchy problem

$$u_{tt} = Lu, x \in \mathbb{R}^n, t \in (0, \infty), \quad (5.41)$$

$$u(x, 0) = f(x), u_t(x, 0) = 0. \quad (5.42)$$

We use everywhere below in Section 5 the following assumption.

**Assumption 5.1.** We assume that in (5.38), (5.40) integers  $k \geq 2, p \geq 4$ , coefficients of the operator  $L$  and the initial condition  $f$  are such that there exists unique solution  $u \in C^4(\mathbb{R}^n \times [0, T])$ ,  $\forall T > 0$  of the problem (5.41), (5.42) satisfying

$$\|u\|_{C^4(\mathbb{R}^n \times [0, T])} \leq B e^{dT}, \forall T > 0, \quad (5.43)$$

where the constants  $B = B(L, \overline{B}) > 0, d = d(L, \overline{B}) > 0$  depend only from the coefficients of the operator  $L$  and the upper estimate  $\overline{B}$  of the norm  $\|f\|_{C^p(\overline{\Omega})} \leq \overline{B}$ .

Note that (5.40) as well as the finite speed of propagation of the solution of problem (5.41), (5.42) guarantee that the function  $u(x, t)$  has a finite support  $\Psi(T) \subset \mathbb{R}^n, \forall t \in (0, T), \forall T > 0$ , see, e.g. the book of Ladyzhenskaya [106]. Hence,  $C^4(\mathbb{R}^n \times [0, T])$  in Assumption 5.1 is actually the space  $C^4(\overline{\Psi(T)} \times [0, T])$ . Using the classical tool of energy estimates [106], one can easily find non-restrictive sufficient conditions imposed on coefficients of the operator  $L$  and the function  $f$  guaranteeing the smoothness  $u \in C^4(\mathbb{R}^n \times [0, T]), \forall T > 0$  as well as (5.43). We are not doing this here for brevity. We consider the following two Inverse Problems.

**Inverse Problem 5.2 (IP5.2).** *Suppose that conditions (5.37)-(5.40) and Assumption 5.1 hold. Let  $u \in C^4(\mathbb{R}^n \times [0, T]), \forall T > 0$  be the solution of the problem (5.41), (5.42). Assume that the function  $f(x)$  is unknown. Determine this function, assuming that the following function  $\varphi_1(x, t)$  is known*

$$u|_{S_\infty} = \varphi_1(x, t). \quad (5.44)$$

**Inverse Problem 5.3 (IP5.3).** *Suppose that conditions (5.37)-(5.40) and Assumption 5.1 hold. Let  $u \in C^4(\mathbb{R}^n \times [0, T]), \forall T > 0$  be the solution of the problem (5.41), (5.42). Assume that the function  $f(x)$  is unknown. Determine this function, assuming that the following function  $\varphi_2(x, t)$  is known*

$$u|_{x \in P_\infty} = \varphi_2(x, t). \quad (5.45)$$

IP5.2 has complete data collection, since the function  $\varphi_1$  is known at the entire boundary of the domain of interest  $\Omega$ . On the other hand, IP5.3 is a special case of incomplete data collection, since  $\Omega \subset \{x_1 > 0\}$ .

In stability estimates one is usually interested to see how the solution varies for a small variation of the input data. Therefore, following (5.43), (5.44) and (5.45), we assume that in the case of IP5.2

$$\|\varphi_1\|_{C^4(\overline{S_T})} \leq \delta e^{dT}, \forall T > 0, \quad (5.46)$$

and in the case of IP5.3

$$\|\varphi_2\|_{C^4(\overline{P_T})} \leq \delta e^{dT}, \forall T > 0, \quad (5.47)$$

where  $\delta \in (0, 1)$  is a sufficiently small number. Note that it is not necessary that  $\delta = B$ , where  $B$  is the number from (5.43). Indeed, while the number  $B$  is involved in the estimate of the norm  $\|u\|_{C^4(\mathbb{R}^n \times [0, T])}, \forall T > 0$  in the entire space, the number  $\delta$  is a part of the estimate of the norm of the boundary data for both IP5.2 and IP5.3.

#### Remarks 5.1.

1. The number  $\delta$  can be viewed as an upper estimate of the level of the error in the data  $\varphi_1, \varphi_2$ . Hence, Theorems 5.3 and 5.4 below address the question of estimating variations of the solution  $f$  of either IP5.2 or IP5.3 via the upper estimate of the level of the error in the data.

2. Since the kernel of the transform  $\mathcal{L}$  in (5.48) decays rapidly with  $\tau \rightarrow \infty$ , then the condition  $t \in (0, \infty)$  in (5.44), (5.45) is not a serious restriction from the applied standpoint. In addition, if having the data in (5.44), (5.45) only on a finite time interval  $t \in (0, T)$  and knowing an upper estimate of a norm of the function  $f$  in (5.42), one can estimate the error in the integral (5.48) when integrating over  $\tau \in (T, \infty)$ . Next, this error can be incorporated in the stability estimates of theorems of this section.

**5.3.2. Transformation to the parabolic case.** Consider the following analog of the Reznickaya's transform (3.38)

$$\mathcal{L}g = \overline{g}(t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{\tau^2}{4t}\right) g(\tau) d\tau. \quad (5.48)$$

The transformation (5.48) is valid for, e.g. all functions  $g \in C[0, \infty)$  which satisfy  $|g(t)| \leq A_g e^{k_g t}$ , where  $A_g$  and  $k_g$  are positive constants depending on  $g$ . It follows from (5.43) that the solution  $u(x, t)$  of the problem (5.41), (5.42) satisfies this condition together with its derivatives up to the fourth order. Obviously

$$\frac{\partial}{\partial t} \left[ \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{\tau^2}{4t}\right) \right] = \frac{\partial^2}{\partial \tau^2} \left[ \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{\tau^2}{4t}\right) \right].$$

Hence,

$$\mathcal{L}(g'') = \overline{g}'(t), \forall g \in C^2[0, \infty) \text{ such that } g'(0) = 0. \quad (5.49)$$

Changing variables in (5.48)  $\tau \Leftrightarrow z, z := \tau/2\sqrt{t}$ , we obtain  $\lim_{t \rightarrow 0+} \overline{g}(t) = g(0)$ . Denote

$$v := \mathcal{L}u. \quad (5.50)$$

It follows from (5.43) and (5.49) that

$$v \in C^{2+\alpha, 1+\alpha/2}(\mathbb{R}^n \times [0, T]), \forall \alpha \in (0, 1), \forall T > 0. \quad (5.51)$$

By (5.41), (5.42) and (5.51) the function  $v(x, t)$  is the solution of the following parabolic Cauchy problem

$$v_t = Lv, x \in \mathbb{R}^n, t > 0, \quad (5.52)$$

$$v(x, 0) = f(x). \quad (5.53)$$

We refer here to the well known uniqueness result for the solution  $v \in C^{2+\alpha, 1+\alpha/2}(\mathbb{R}^n \times [0, T]), \forall T > 0$  of the problem (5.52), (5.53), see, e.g. the book of Ladyzhenskaya, Solonnikov and Uralceva [104].

Below in Sections 5.3.2 and 5.3.3 we work only with the function  $v$ . Thus, we set everywhere below  $T := 1$  for the sake of definiteness. Denote  $S_1 = \partial\Omega \times (0, 1)$ ,  $P_1 = P \times (0, 1)$ ,

$$\mathcal{L}\varphi_1 := \overline{\varphi}_1(x, t) = v|_{S_1}, \mathcal{L}\varphi_2 := \overline{\varphi}_2(x, t) = v|_{P_1}. \quad (5.54)$$

Then

$$\overline{\varphi}_1 \in C^{2+\alpha, 1+\alpha/2}(\overline{S}_1), \overline{\varphi}_2 \in C^{2+\alpha, 1+\alpha/2}(\overline{P}_1). \quad (5.55)$$

Let

$$\overline{\psi}_1(x, t) = \partial_\nu v|_{S_1}, \overline{\psi}_2(x, t) = \partial_{x_1} v|_{P_1}. \quad (5.56)$$

By Theorem 5.2 of Chapter IV of [104], (5.43) and (5.54)-(5.56) there exist numbers  $C(\Omega, L), C(P, L) > 0$  depending only on listed parameters such that

$$\|\overline{\psi}_1\|_{C^{1+\alpha, \alpha/2}(\overline{S}_1)} \leq C(\Omega, L) \|\overline{\varphi}_1\|_{C^{2+\alpha, 1+\alpha/2}(\overline{S}_1)}, \quad (5.57)$$

$$\|\overline{\psi}_2\|_{C^{1+\alpha, \alpha/2}(\overline{P}_1)} \leq C(P, L) \|\overline{\varphi}_2\|_{C^{2+\alpha, 1+\alpha/2}(\overline{P}_1)}. \quad (5.58)$$

We now describe an elementary and well known procedure of finding the normal derivative of the function  $v$  either at  $S_1$  (in the case of IP5.2) or at  $P_1$  (in the case of IP5.3). In fact, an analog of this procedure was described in Section 5.2 for the hyperbolic case, see (5.32), (5.33). In the case of IP5.2 we solve the initial boundary value problem for equation (5.52) for  $(x, t) \in (\mathbb{R}^n \setminus \Omega) \times (0, 1)$  with the zero initial condition in  $\mathbb{R}^n \setminus \Omega$  (because of (5.40)) and the Dirichlet boundary condition  $v|_{S_1} = \overline{\varphi}_1$ . Then we uniquely find the normal derivative  $\partial_\nu v|_{S_1} = \overline{\psi}_1$ . Similarly, in the case of IP5.3, we uniquely find the Neumann boundary condition  $\partial_{x_1} v|_{P_1} = \overline{\psi}_2$ . Estimates (5.57), (5.58) ensure the stability of this procedure.

Therefore, problems IP5.2 and IP5.3 are replaced with a corresponding inverse problem for the parabolic PDE (5.52) with the lateral Cauchy data (5.54), (5.56). These data are given at  $S_1$  for IP5.2 and at  $P_1$  for IP5.3. Uniqueness of the solution of each of these parabolic inverse problems follows from Theorem 2.2 and Remark 2.2.

Using (5.46), (5.48), (5.49) and (5.54)-(5.58), we obtain

$$\|\overline{\varphi}_1\|_{C^{2+\alpha, 1+\alpha/2}(\overline{S}_1)} + \|\overline{\psi}_1\|_{C^{1+\alpha, \alpha/2}(\overline{S}_1)} \leq C_1(\Omega, L, d) \delta. \quad (5.59)$$

Next, using (5.47) instead of (5.46), we obtain

$$\|\overline{\varphi}_2\|_{C^{2+\alpha, 1+\alpha/2}(\overline{P}_1)} + \|\overline{\psi}_2\|_{C^{1+\alpha, \alpha/2}(\overline{P}_1)} \leq C_2(P, L, d) \delta, \quad (5.60)$$

where constants  $C_1(\Omega, L, d), C_2(P, L, d) > 0$  depend only on listed parameters. It follows from (5.59) that with a different constant  $\overline{C} := \overline{C}(\Omega, L, d) > 0$

$$\|\overline{\varphi}_1\|_{H^1(S_1)} + \|\overline{\psi}_1\|_{L_2(S_1)} \leq \overline{C} \delta. \quad (5.61)$$

**5.3.3. Logarithmic stability estimates for IP5.2 and IP5.3. Theorem 5.4.** *Consider IP5.2. Let Assumption 5.1 holds and conditions (5.40), (5.46) be valid. Also, assume that the upper bound  $F$  of the norm  $\|\nabla f\|_{L_2(\Omega)}$  is given,  $\|\nabla f\|_{L_2(\Omega)} \leq F$ . Then there exists a sufficiently small number  $\delta_0 = \delta_0(L, \Omega) \in (0, 1)$  and a constant  $M_1 = M_1(L, \Omega) > 0$ , both dependent only on listed parameters, such that if the number  $\delta$  in (5.46) is so small that  $\bar{C}\delta \in (0, \delta_0)$ , then the following logarithmic stability estimate holds*

$$\|f\|_{L_2(\Omega)} \leq \frac{M_1 F}{\sqrt{\ln \left[ (\bar{C}\delta)^{-1} \right]}}.$$

Here  $\bar{C} = \bar{C}(\Omega, L, d) > 0$  is the number in (5.61).

Consider now IP5.3. Denote  $\bar{x} = (x_2, \dots, x_n)$ . Changing variables  $(x', t') = (\sqrt{b}x, dt)$  with an appropriate constant  $b > 0$  and keeping the same notations for new variables for brevity, we obtain that without loss of generality we can assume that

$$\Omega \subset \left\{ x_1 + |\bar{x}|^2 < \frac{1}{4}, x_1 > 0 \right\}. \quad (5.62)$$

Denote

$$\Phi = \left\{ (x, t) : x_1 \in (0, 1), \bar{x} = (x_2, x_3, \dots, x_n) \in (-1, 1)^{n-1}, t \in (0, 1) \right\}, \quad (5.63)$$

$$\partial_1 \Phi = \bar{\Phi} \cap P = \left\{ (x, t) : x_1 = 0, \bar{x} \in (-1, 1)^{n-1}, t \in (0, 1) \right\}. \quad (5.64)$$

Recall that (5.47) implies (5.60). Hence, assuming that (5.47) holds and using (5.63), (5.64), we derive that there exists a constant  $\tilde{C} = \tilde{C}(L, \Phi, d) > 0$  such that

$$\|\bar{\varphi}_2\|_{H^1(\partial_1 \Phi)} + \|\bar{\psi}_2\|_{L_2(\partial_1 \Phi)} \leq \tilde{C}\delta. \quad (5.65)$$

**Theorem 5.5.** *Consider IP5.3. Let Assumption 5.1 holds and (5.40), (5.47) be valid. Also, assume that for a certain  $\alpha \in (0, 1)$  the upper bound  $F$  of the norm  $\|f\|_{C^{2+\alpha}(\bar{\Omega})}$  is given, i.e.  $\|f\|_{C^{2+\alpha}(\bar{\Omega})} \leq F$ . Then there exists a sufficiently small number  $\delta_0 = \delta_0(L, \Phi) \in (0, 1)$  and a constant  $M_2 = M_2(L, \Phi) > 0$ , both dependent only on listed parameters, such that if the number  $\delta$  in (5.47) is so small that  $\tilde{C}\delta \in (0, \delta_0)$ , then the following logarithmic stability estimate holds*

$$\|f\|_{L_2(\Omega)} \leq \frac{M_2 F}{\sqrt{\ln \left[ (\tilde{C}\delta)^{-1} \right]}}.$$

Here  $\tilde{C} = \tilde{C}(L, \Phi, d) > 0$  is the number from (5.65).

**5.4. QRM in the hyperbolic case.** Although the Quasi-Reversibility Method (QRM) was discussed in Section 2.5, it is worth to discuss it here again for the specific hyperbolic case. The reason is that we now have two types of stability estimates, which are different from the Hölder stability estimate of Section 2.5: the Lipschitz stability estimates of Theorems 5.1, 5.2 and the logarithmic stability estimates of Theorems 5.3, 5.4. We start from IP5.3. The case of IP 5.2 is not discussed here since it is similar.

**5.4.1. QRM for IP5.3.** To use embedding theorem, we work in this section only in 3d. The 2d case is similar. We now increase the required smoothness of the solution of the problem (5.41), (5.42). To do this, we replace Assumption 5.1 with Assumption 5.2, where we use the norm  $\|u\|_{C^8(\mathbb{R}^n \times [0, T])}$  instead of the norm  $\|u\|_{C^4(\mathbb{R}^n \times [0, T])}$  of Assumption 5.1.

**Assumption 5.2.** We assume that integers  $k, p$  in (5.38), (5.40), coefficients of the operator  $L$  and the initial condition  $f$  are such that there exists unique solution  $u \in C^{12}(\mathbb{R}^n \times [0, T])$ ,  $\forall T > 0$  of the problem (5.41), (5.42) satisfying

$$\|u\|_{C^{12}(\mathbb{R}^n \times [0, T])} \leq B e^{dT}, \forall T > 0, \quad (5.66)$$

where the constants  $B = B(L, \overline{B}) > 0, d = d(L, \overline{B}) > 0$  depend only from the coefficients of the operator  $L$  and an upper estimate  $\overline{B}$  of the norm  $\|f\|_{C^p(\overline{\Omega})} \leq \overline{B}$ .

We assume in Section 5.4.1 that Assumption 5.2 holds. Hence, Theorem 5.2 of Chapter IV of the book [104], (5.66) and (5.54)-(5.56) imply that functions

$$\overline{\varphi}_2 \in C^{10+\alpha, 5+\alpha/2}(\overline{P}_1), \overline{\psi}_2 \in C^{8+\alpha, 4+\alpha/2}(\overline{P}_1) \quad (5.67)$$

and there exists a number  $C_3(P, L) > 0$  depending only on listed parameters such that

$$\|\overline{\psi}_2\|_{C^{8+\alpha, 4+\alpha/2}(\overline{P}_1)} \leq C_3(P, L) \|\overline{\varphi}_2\|_{C^{10+\alpha, 5+\alpha/2}(\overline{P}_1)}. \quad (5.68)$$

Let sets  $\Phi, \partial_1 \Phi$  be the ones introduced in (5.63), (5.64). Then (5.67) and (5.68) imply that

$$\overline{\varphi}_2, \overline{\psi}_2 \in H^{8,4}(\partial_1 \Phi), \|\overline{\varphi}_2\|_{H^{8,4}(\partial_1 \Phi)} + \|\overline{\psi}_2\|_{H^{8,4}(\partial_1 \Phi)} \leq C_4(P, \Phi, L) \|\overline{\varphi}_2\|_{C^{10+\alpha, 5+\alpha/2}(\overline{P}_1)}, \quad (5.69)$$

where the number  $C_4(P, \Phi, L) = \text{const.} > 0$  depends only on listed parameters.

Let the function  $v(x, t)$  be the one defined in (5.50). Then  $v \in C^{10+\alpha, 5+\alpha/2}(\mathbb{R}^n \times [0, T])$ ,  $\forall T > 0$  is the solution of the problem (5.52), (5.53). Denote

$$r(x, t) = \overline{\varphi}_2(x, t) + x_1 \overline{\psi}_2(x, t) = \overline{\varphi}_2(\overline{x}, t) + x_1 \overline{\psi}_2(\overline{x}, t), \quad (5.70)$$

$$\widehat{v}(x, t) = v(x, t) - r(x, t), \quad (5.71)$$

$$p(x, t) = -(r_t - Lr)(x, t). \quad (5.72)$$

Let  $H_0^4(\Phi) := \{u \in H^4(\Phi) : u|_{\partial_1 \Phi} = u_{x_1}|_{\partial_1 \Phi} = 0\}$ . Then

$$\widehat{v}_t - L\widehat{v} = p(x, t), (x, t) \in \Phi, \widehat{v} \in H_0^4(\Phi), \quad (5.73)$$

$$\widehat{v}|_{\partial_1 \Phi} = 0, \widehat{v}_{x_1}|_{\partial_1 \Phi} = 0. \quad (5.74)$$

To solve IP5.3 via the QRM, we minimize the following Tikhonov functional

$$J_\gamma(\widehat{v}) = \|\widehat{v}_t - L\widehat{v} - p\|_{L_2(\Phi)}^2 + \gamma \|\widehat{v}\|_{H^4(\Phi)}^2, \widehat{v} \in H_0^4(\Phi), \quad (5.75)$$

where  $\gamma > 0$  is the regularization parameter. The requirement  $\widehat{v} \in H^4(\Phi)$  is an over-smoothness. This condition is imposed to ensure that the function  $\widehat{v} \in C^1(\overline{\Phi})$ : because of the embedding theorem. Indeed, we need the smoothness  $\widehat{v} \in C^1(\overline{\Phi})$  to apply theorems 2.2 and 3.1 of [95]. We imposed in Assumption 4.2  $u \in C^{12}(\mathbb{R}^n \times [0, T])$ ,  $\forall T > 0$  only to guarantee that  $\widehat{v} \in H^4(\Phi) \subset C^1(\overline{\Phi})$ . However, the author's numerical experience with QRM has consistently demonstrated that one can significantly relax the required smoothness in practical computation, see [91, 100, 101]. This is likely because one is not using an overly small grid step size in finite differences when minimizing functionals like the one in (5.75). Hence, one effectively works with a finite dimensional space with not too many dimensions. This means that one can rely in this case on the equivalence of all norms in finite dimensional spaces. Thus, most likely one can replace in real computations  $\gamma \|v\|_{H^4(\Phi)}^2$  with  $\gamma \|v\|_{H^{2,1}(\Phi)}^2$ .

Let  $(\cdot, \cdot)$  and  $[\cdot, \cdot]$  be scalar products in  $L_2(\Phi)$  and  $H^4(\Phi)$  respectively. Let the function  $u_\gamma \in H_0^4(\Phi)$  be a minimizer of the functional (5.75). Then the variational principle implies that

$$(\partial_t u_\gamma - Lu, \partial_t w - Lw) + \gamma [u, w] = (p, w_t - Lw), \forall w \in H_0^4(\Phi).$$

**Lemma 5.1.** *For every function  $p \in L_2(\Phi)$  and every  $\gamma > 0$  there exists unique minimizer  $u_\gamma = u_\gamma(p) \in H_0^4(\Phi)$  of the functional (5.75). Furthermore, there exists a constant  $M = M(L, \Phi)$  such that the following estimate holds  $\|u_\gamma\|_{H^4(\Phi)} \leq M_1 \gamma^{-1/2} \|p\|_{L_2(\Phi)}$ .*

Lemma 5.1 is an obvious analog of Lemma 2.5. The idea now is that if  $u_\gamma(x, t) \in H_0^4(\Phi)$  is the minimizer mentioned in Lemma 5.1, then the approximate solution of IP5.3 is

$$f_\gamma(x) = u_\gamma(x, 0) + r(x, 0). \quad (5.76)$$

The question of convergence of minimizers of  $J_\gamma$  to the exact solution is more difficult than the existence question of Lemma 5.1. To address the question of convergence, we need first to introduce the exact solution as well as the error in the data, just as this is always done in the regularization theory [5, 14, 49, 70, 145], also, see Theorem 2.7. We assume that there exists an “ideal” noiseless data  $\varphi_2^* \in C^{12}(P_\infty)$ . Following (5.47) and (5.66), we assume that with a sufficiently small number  $\delta \in (0, 1)$  the following estimate holds

$$\|\varphi_2 - \varphi_2^*\|_{C^{12}(\overline{P}_T)} \leq \delta e^{dT}, \forall T > 0. \quad (5.77)$$

Let  $\overline{\varphi}_2^* = \mathcal{L}\varphi_2^*$ , the function  $f^*$  satisfying (5.40) is the solution of IP5.3 for the case of the noiseless data  $\varphi_2^*$ , the function  $u^* \in C^{12}(\mathbb{R}^n \times [0, T])$ ,  $\forall T > 0$  satisfying (5.66) is the solution of the Cauchy problem (5.41), (5.42) with  $f := f^*$ . Following (5.50), denote  $v^* = \mathcal{L}u^*$ . Let functions  $\overline{\varphi}_2^*, \dots, p^*$  have the same meaning as corresponding functions in (5.67)-(5.72), except that they are generated by the noiseless data  $\varphi_2^*$ . Then (5.69) and (5.77) imply that

$$\|p - p^*\|_{L_2(\Phi)} \leq C_5(\Phi, L, d) \delta, \quad (5.78)$$

where the constant  $C_5(P, \Phi, L, d) > 0$  depends only on listed parameters.

Theorem 5.6 establishes the convergence rate of the QRM. The proof of this theorem is using (5.78) and is similar with the proof of Theorem 3.1 of [95].

**Theorem 5.6.** *Let Assumption 5.2 and condition (5.77) be valid. Suppose that the regularization parameter  $\gamma = \gamma(\delta) := \delta \in (0, 1)$ . Let the function  $u_\gamma \in H_0^4(\Phi)$  be the unique minimizer of the functional (5.75) (Lemma 5.1). Then there exists a number  $Y = Y(\Phi, L, B, d) > 0$  and a sufficiently small number  $\delta_0 = \delta_0(L, \Phi) \in (0, 1)$  such that if  $\delta \in (0, \delta_0)$ , then the following logarithmic convergence rate takes place*

$$\|f_{\gamma(\delta)} - f^*\|_{L_2(\Omega)} \leq \frac{Y}{\sqrt{\ln(\delta^{-1})}}, \quad (5.79)$$

where the function  $f_{\gamma(\delta)}(x)$  is defined in (5.76). In addition, for every  $\omega \in (0, \omega_0)$  there exists a number  $\rho = \rho(L, \Phi, B, d) \in (0, 1/2)$  such that the following convergence rate takes place

$$\|u_{\gamma(\eta)} - \widehat{v}^*\|_{H^{1,0}(D_{1/2})} \leq Y \delta^\rho,$$

where the domain  $D_{1/2} = \{(x, t) : x_1 > 0, x_1 + |\overline{x}|^2 + (t - 1/2)^2 < 1/4\}$ .

**5.4.2. QRM for Problem 5.1 of Section 5.1.** In this section we assume that condition (5.6) is in place. This enables us to use the Lipschitz stability estimate (5.10) of Theorem 5.1. We consider the problem of the recovery of the function  $u(x, t)$  from conditions (5.3), (5.4). We rewrite these conditions now as

$$Au := u_{tt} - c^2(x) \Delta u - \sum_{j=1}^{n+1} b_j(x, t) u_{x_j} - b_0(x, t) u = f(x, t), (x, t) \in Q_T^\pm, \quad (5.80)$$

$$u|_{S_T^\pm} = p(x, t), \partial_n u|_{S_T^\pm} = q(x, t), \quad (5.81)$$

where  $u_{n+1} := u_t$ . Suppose that there exists a function  $F(x, t)$  such that

$$F \in H^2(Q_T^\pm), F|_{S_T^\pm} = p(x, t), \partial_n F|_{S_T^\pm} = q(x, t).$$



Denote  $w = u - F, G = f - AF$ . Let  $H_0^2(Q_T^\pm) = \{U \in H^2(Q_T^\pm) : U|_{S_T^\pm} = \partial_n U|_{S_T^\pm} = 0\}$ . Using (5.80) and (5.81), we obtain

$$Aw = G, (x, t) \in Q_T^\pm, w \in H_0^2(Q_T^\pm). \quad (5.82)$$

Thus, now we are concerned with finding the function  $w$  satisfying (5.82). QRM for this problem amounts to the minimization of the following Tikhonov functional

$$V_\gamma(w) = \|Aw - G\|_{L_2(Q_T^\pm)}^2 + \gamma \|w\|_{H^2(Q_T^\pm)}^2, w \in H_0^2(Q_T^\pm). \quad (5.83)$$

As usual,  $\gamma > 0$  is the regularization parameter here. Let  $w_\gamma \in H_0^2(Q_T^\pm)$  be a minimizer of the functional (5.83),  $(\cdot, \cdot)$  be the scalar product in  $L_2(Q_T^\pm)$  and  $[\cdot, \cdot]$  be the scalar product in  $H^2(Q_T^\pm)$ . Then the variational principle implies that

$$(Aw_\gamma, Av) + \gamma [w_\gamma, v] = (G, Av), \forall v \in H_0^2(Q_T^\pm). \quad (5.84)$$

Lemma 5.2 is again an obvious analog of Lemma 2.5.

**Lemma 5.2.** *Let the function  $G \in L_2(Q_T^\pm)$ . Then for every  $\gamma > 0$  there exists unique minimizer  $w_\gamma \in H_0^2(Q_T^\pm)$  of the functional (5.84). Furthermore, with a constant  $C_6 = C_6(A, Q_T^\pm) > 0$  depending only on listed parameters the following estimate holds*

$$\|w_\gamma\|_{H^2(Q_T^\pm)} \leq \frac{C}{\sqrt{\gamma}} \|G\|_{L_2(Q_T^\pm)}.$$

**Theorem 5.7** (convergence). *Let the domain  $\Omega = \{|x| < R\}$ . Assume that the coefficient  $c \in C^1(\overline{\Omega})$  of the operator  $A$  in (5.80) satisfies conditions (5.6), (5.8) and other coefficients of the operator  $A$  are such that  $b_j \in C(\overline{Q_T^\pm})$ ,  $j \in [0, n+1]$ . Let  $T > R/\sqrt{\eta_0}$ , where the constant  $\eta_0 = \eta_0(R, d, \alpha) \in (0, 1]$  was defined in Theorem 5.1. Assume that there exists exact solution  $w^* \in H_0^2(Q_T^\pm)$  of the problem (5.82) with the exact data  $G^*$ . Let  $w_\gamma \in H_0^2(Q_T^\pm)$  be the unique minimizer of the functional (5.83) (Lemma 5.2). Then there exists a constant  $K = K(A, R, T, d, \alpha) > 0$  such that*

$$\|w_\gamma - w^*\|_{H^1(Q_T^\pm)} \leq K \left( \|G - G^*\|_{L_2(Q_T^\pm)} + \sqrt{\gamma} \|w^*\|_{H^2(Q_T^\pm)} \right). \quad (5.85)$$

In particular, if  $c(x) \equiv 1$ , then it is sufficient to have  $T > R$ . Also, if  $\|G - G^*\|_{L_2(Q_T^\pm)} \leq \delta$ , where  $\delta \in (0, 1)$  is the level of the error in the data, and if  $\gamma \in (0, \delta^2]$ , then  $\|w_\gamma - w^*\|_{H^1(Q_T^\pm)} \leq K\delta$ .

**Proof.** We have

$$(Aw^*, Av) + \gamma [w^*, v] = (G^*, Av) + \gamma [w^*, v], \forall v \in H_0^2(Q_T^\pm).$$

Subtract this equality from (5.84) and denote  $\tilde{w} = w_\gamma - w^*$ ,  $\tilde{G} = G - G^*$ . Then  $\tilde{w} \in H_0^2(Q_T^\pm)$  and

$$(A\tilde{w}, Av) + \gamma [\tilde{w}, v] = (\tilde{G}, Av) - \gamma [w^*, v], \forall v \in H_0^2(Q_T^\pm).$$

Hence, Cauchy-Bunyakovsky inequality implies that

$$\|A\tilde{w}\|_{L_2(Q_T^\pm)}^2 + \gamma \|\tilde{w}\|_{H^2(Q_T^\pm)}^2 \leq \|\tilde{G}\|_{L_2(Q_T^\pm)}^2 + \gamma \|w^*\|_{H^2(Q_T^\pm)}^2. \quad (5.86)$$

Applying Theorem 5.2 to (5.86), we obtain (5.85).  $\square$



**5.5. Published results.** Stability estimates and convergent numerical methods for Problem 5.1, Problem 5.2, IP5.1, IP5.2 and IP5.3 with an arbitrary time independent principal part of the operator  $L$  in (5.37) were not obtained prior to the work [95]. The Lipschitz stability estimate (5.10) is important for the control theory, since it is used for proofs of exact controllability theorems. For the first time, estimate (5.10) was proved in 1986 by Lop Fat Ho [121] for the equation  $u_{tt} - \Delta u = 0$  with the aim of applying to the control theory. However, the method of multipliers of [121] cannot handle neither variable lower order terms of the operator  $L$  nor a variable coefficient  $c(x)$ . On the other hand, Carleman estimates are not sensitive to lower order terms of PDE operators and also can handle the case of a variable coefficient  $c(x)$ .

For the first time, the Carleman estimate was applied to this problem by Klivanov and Malinsky [81]. In [81] Theorem 5.1 for equation (5.3) with the lateral Cauchy data (5.4) and  $c(x) \equiv 1$  was proved. Using (5.10), an analog of Theorem 5.7 was also proved in [81]. Next, the result of [81] was extended by Kazemi and Klivanov [72] and also by the author in [87] to a more general case of the hyperbolic inequality (5.5) with  $c(x) \equiv 1$ . Although in publications [72, 81, 87]  $c \equiv 1$ , it is clear from them that the key idea is in applying the Carleman estimate, while a specific form of the principal part of the hyperbolic operator is less important. This thought is reflected in the proof of Theorem 3.4.8 of the book of Isakov [67]. Thus, Theorem 5.1 for the variable coefficient  $c(x)$  satisfying an analog of (5.7) was obtained in section 2.4 of the book of Klivanov and Timonov [86] as well as in the paper of Clason and Klivanov [44]. The idea of [72] was used in the control theory by Lasiecka, Triggiani and Yao [107], Lasiecka, Triggiani and Zhang [108, 109, 110] and by Triggiani and Yao [146]. Isakov and Yamamoto [64] used that idea to prove a stronger version of Theorem 5.1.

To prove Lipschitz stability without condition (5.6), one can impose some conditions of Riemannian geometry on the principal part of the operator  $L$  in (5.37), see Bardos, Lebeau and Rauch [6], Lasiecka, Triggiani and Yao [107, 146], Lasiecka, Triggiani and Zhang [108, 109, 110], Romanov [137, 138] and Stefanov and Uhlmann [142]. In [107, 108, 109, 110, 137, 138, 146] Carleman estimates were used.

Analog of Theorem 5.7 about convergence of QRM were proved in Klivanov and Rakesh [83], Clason and Klivanov [44] and Klivanov, Kuzhuget, Kabanikhin and Nechaev [91]. Numerical testing of QRM was performed in these references. This testing has consistently demonstrated a high degree of robustness. For example, accurate results were obtained in [91] with up to 50% noise in the data.

For explicit formulas for the reconstruction of the function  $f(x)$  for TAT (IP5.1) in the case when in (5.37)  $L \equiv \Delta$  we refer to Finch, Patch and Rakesh [51], Finch, Haltmeier and Rakesh [52], the review paper of Kuchment and Kunyansky [97] and Kunyansky [98]. These formulas lead to some stability estimates as well as to numerical methods with good performances. Another numerical method for TAT was proposed by Agranovsky and Kuchment [2].

**6. Approximately Globally Convergent Numerical Method.** The first step of the numerical method outlined in this section is the elimination of the unknown coefficient from the underlying PDE, which is the same step as in BK. In this section we briefly outline the recently developed numerical method of Beilina and Klivanov referring for details to [4, 14, 15, 16, 17, 18, 19, 20, 21, 92, 93, 100, 101, 102, 103]. Numerical tests are not presented here, since they are published in these works.

Even though the field of Inverse Problems is an applied one and even though MCIPs have been studied by many researchers since 1960-ies, the topic of reliable numerical methods for them is still in its infancy. This is because of *enormous challenges* one inevitably faces when trying to study this topic. Those challenges are caused by two factors combined: nonlinearity and ill-posedness of MCIPs. In the case of single measurement the third complicating factor is the minimal amount of available information. Conventional least squares functionals for MCIPs suffer from the phenomenon of multiple local minima and ravines. This leads to locally convergent numerical methods, which require a good first guess about the solution. However, the latter is impractical.

In the above cited series of recent publications some properties of underlying PDE operators instead of least squares functionals were used. A *very important* feature of this numerical method is that it does not require any knowledge of neither the medium inside of the domain of interest nor of any point in a small neighborhood of the true solution. For the first time, the following two goals were *simultaneously* achieved

for MCIPs for a hyperbolic PDE with single measurement data:

**Goal 1.** The development of such a numerical method, which would have a rigorous guarantee of obtaining at least one point in a small neighborhood of the exact solution without any advanced knowledge of that neighborhood.

**Goal 2.** This numerical method should have a good performance on computationally simulated data. In addition, if experimental data are available, then this method should demonstrate a good performance on these data.

A *crucial requirement* is to achieve both these goals *simultaneously* rather than just only one of them. Because of the above mentioned substantial challenges, it is natural to have the rigorous guarantee of Goal 1 within the framework of a reasonable approximate mathematical model. Since convergence is guaranteed in the framework of that model, then we call our numerical method *approximately globally convergent*. In principle, any mathematical model can be called “approximate”. Therefore, the validity of our model is verified via a six-step procedure, see [14, 21]. Basically this procedure includes the proof of a convergence theorem and computational results for both synthetic and experimental data. If convergence theorem satisfying Goal 1 is proved and computational results are good ones (Goal 2), especially ones for experimental data, then that approximate mathematical model is proclaimed as a valid one.

On the other hand, nothing works without such an approximate model. Indeed, the author is unaware about such numerical methods for MCIPs with single measurement data, which would: (1) simultaneously achieve Goals 1 and 2 and, at the same time, (2) would not rely on some reasonable approximations, which cannot be rigorously justified.

**6.1. Outline of the method.** Let  $\Omega \subset \mathbb{R}^3$  be a convex bounded domain with the boundary  $\partial\Omega \in C^3$ . Let  $d = \text{const.} > 2$ . We assume that the coefficient  $c(x)$  satisfies the following conditions

$$c(x) \in [1, d], \quad c(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega, \quad c \in C^\alpha(\mathbb{R}^3). \quad (6.1)$$

We assume *a priori* knowledge of the constant  $d$ , which amounts to the knowledge of the correctness set in the theory of Ill-Posed problems [5, 14, 49, 70, 145]. Consider the Cauchy problem for the hyperbolic equation

$$c(x) u_{tt} = \Delta u \text{ in } \mathbb{R}^3 \times (0, \infty), \quad (6.2)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = \delta(x - x_0). \quad (6.3)$$

A similar technique can be developed for MCIPs for the parabolic equation  $c(x) v_t = \Delta v + a(x) v$ , where either of coefficients  $c(x)$  or  $a(x)$  is unknown [14]. Equation (6.2) governs, e.g. propagation of acoustic and electromagnetic waves. In the acoustical case  $c(x) = b^{-2}(x)$ , where  $b(x)$  is the sound speed. In the 2-D case of EM waves propagation, the dimensionless coefficient is  $c(x) = \varepsilon_r(x)$ , where  $\varepsilon_r(x)$  is the spatially distributed dielectric constant of the medium. In the latter case the assumption  $c(x) = 1$  for  $x \in \mathbb{R}^3 \setminus \Omega$  in (6.1) means that we have air outside the medium of interest  $\Omega$ . And the assumption  $c(x) \geq 1$  reflects the fact that the dielectric constants of almost all materials exceed the one of the air. Equation (6.2) was successfully used in [14, 18, 92] to work with experimental data, which are obviously in 3d. The latter was recently explained by Beilina in [22]. It was shown in Test 4 of [22] that the component of the electric field  $E(x, t) = (E_1, E_2, E_3)(x, t)$ , which was originally initialized, strongly dominates two other components.

**Multidimensional Coefficient Inverse Problem (MCIP).** Assume that the coefficient  $c(x)$  of equation (6.2) satisfies condition (6.1) and is unknown in the domain  $\Omega$ . Determine the function  $c(x)$  for  $x \in \Omega$ , assuming that the following function  $g(x, t)$  is known for a single source position  $x_0 \notin \overline{\Omega}$  in (6.3)

$$u(x, t) = g(x, t), \quad \forall (x, t) \in \partial\Omega \times (0, \infty). \quad (6.4)$$

The function  $g(x, t)$  models time dependent measurements of the wave field at the boundary of the domain of interest. The assumption of the infinite time interval in (6.4) is not a restrictive one, because we work with the Laplace transform of the function  $u(x, t)$ , and the kernel of this transform decays rapidly as

$t \rightarrow \infty$ . In this MCIP the data  $g(x, t)$  are assumed to be known at the entire boundary  $\partial\Omega$ . The case of backscattering data can be treated similarly, see Chapter 6 in the book [14] as well as [21, 102, 103].

Consider the Laplace transform of the function  $u$ ,

$$w(x, s) = \int_0^\infty u(x, t) e^{-st} dt, \text{ for } s > \underline{s} = \text{const.} > 0. \quad (6.5)$$

We assume that the number  $\underline{s}$  is sufficiently large, so that the integral (6.5) converges absolutely and that the same is valid for the derivatives  $D^k u, k = 0, 1, 2$ . We call the parameter  $s$  *pseudo frequency*. It can be proven that

$$\Delta w - s^2 c(x) w = -\delta(x - x_0), \quad x \in \mathbb{R}^3, \quad (6.6)$$

$$\lim_{|x| \rightarrow \infty} w(x, s) = 0. \quad (6.7)$$

Furthermore, for each value of  $s > 0$  the problem (6.6), (6.7) has unique solution of the form [14, 21]

$$w(x, s) = \frac{\exp(-s|x - x_0|)}{4\pi|x - x_0|} + \overline{w}(x, s), \quad \overline{w} \in C^{2+\alpha}(\mathbb{R}^3), \quad \lim_{|x| \rightarrow \infty} \overline{w}(x, s) = 0. \quad (6.8)$$

Suppose that geodesic lines generated by the function  $c(x)$  are regular and  $c(x)$  is sufficiently smooth. Let  $\tau(x, x_0)$  be the length of the geodesic line connecting points  $x$  and  $x_0$ . Then

$$|D_s^k w(x, s)|_{2+\alpha} = \left| D_s^k \left\{ \frac{\exp[-s\tau(x, x_0)]}{f(x, x_0)} \right\} \right|_{2+\alpha} \left[ 1 + O\left(\frac{1}{s}\right) \right], \quad s \rightarrow \infty, k = 0, 1, \quad (6.9)$$

where  $|\cdot|_{2+\alpha} = \|\cdot\|_{C^{2+\alpha}(\overline{\Omega})}$ ,  $f(x, x_0)$  is a certain function and  $f(x, x_0) \neq 0$  for  $x \in \overline{\Omega}$ . It is unclear how to effectively verify the regularity of geodesic lines for generic functions  $c(x)$ . Therefore, we assume below the asymptotic behavior (6.9) without linking it to the regularity of geodesic lines.

We have  $w(x, s) > 0$  [14, 21]. Denote

$$v(x, s) := \frac{\ln w(x, s)}{s^2}.$$

Since the source  $x_0 \notin \overline{\Omega}$ , then

$$\Delta v + s^2 (\nabla v)^2 = c(x), \quad x \in \Omega. \quad (6.10)$$

Now we make the same step as the first step of BK. Differentiate both sides of (6.10) with respect to  $s$ . Let  $q(x, s) = \partial_s v(x, s)$ . Then

$$v(x, s) = - \int_s^{\overline{s}} q(x, \tau) d\tau + V(x, \overline{s}), \quad (6.11)$$

$$V(x, \overline{s}) = v(x, \overline{s}) = \frac{\ln w(x, \overline{s})}{\overline{s}^2}. \quad (6.12)$$

Here the truncation pseudo frequency  $\overline{s} > \underline{s}$  is a large number. We call  $V(x, \overline{s})$  the *tail function*. The tail function is unknown. By (6.9)

$$|V(x, \overline{s})|_{2+\alpha} = O(\overline{s}^{-1}), \quad |\partial_{\overline{s}} V(x, \overline{s})|_{2+\alpha} = O(\overline{s}^{-2}), \quad \overline{s} \rightarrow \infty. \quad (6.13)$$

The number  $\overline{s}$  is the main regularization parameter of our numerical method. In the computational practice  $\overline{s}$  is chosen in numerical experiments.

Thus, we obtain from (6.10), (6.11) the following nonlinear integral differential equation

$$\begin{aligned} \Delta q - 2s^2 \nabla q \int_s^{\bar{s}} \nabla q(x, \tau) d\tau + 2s \left[ \int_s^{\bar{s}} \nabla q(x, \tau) d\tau \right]^2 \\ + 2s^2 \nabla q \nabla V - 4s \nabla V \int_s^{\bar{s}} \nabla q(x, \tau) d\tau + 2s (\nabla V)^2 = 0, x \in \Omega. \end{aligned} \quad (6.14)$$

It follows from (6.4) that

$$q(x, s) = \psi(x, s), \quad \forall (x, s) \in \partial\Omega \times [\underline{s}, \bar{s}], \quad (6.15)$$

where  $\psi(x, s) = s^{-2} \partial_s \ln \varphi - 2s^{-3} \ln \varphi$  and  $\varphi(x, s)$  is the Laplace transform (6.5) of the function  $g(x, t)$  in (6.4). We have two unknown functions  $q$  and  $V$  in equation (6.14). Therefore, to approximate both of them, we approximate the function  $q$  via “inner” iterations and the function  $V$  is approximated via “outer” iterations. Suppose for a moment that functions  $q$  and  $V$  are approximated in  $\Omega$  together with their derivatives  $D_x^\beta q, D_x^\beta V, |\beta| \leq 2$ . Then the corresponding approximation for the target coefficient can be found via (6.10) as

$$c(x) = \Delta v + \underline{s}^2 (\nabla v)^2, x \in \Omega, \quad (6.16)$$

where the function  $v$  is approximated via (6.11). We have found in our numerical experiments that the optimal value of  $s$  to use in (6.16) is  $s := \underline{s}$ .

To solve the problem (6.14), (6.15), we assume that  $q(x, s)$  is a piecewise constant function with respect  $s$ . Hence, we assume that there exists a partition  $\underline{s} = s_N < s_{N-1} < \dots < s_1 < s_0 = \bar{s}, s_{i-1} - s_i = h$  of the interval  $[\underline{s}, \bar{s}]$  with a sufficiently small grid step size  $h$  such that

$$q(x, s) = q_n(x) \text{ for } s \in (s_n, s_{n-1}], q_0 \equiv 0.$$

We approximate the boundary condition (6.15) as a piecewise constant function,  $q_n(x) := \overline{\psi}_n(x), x \in \partial\Omega$ , where  $\overline{\psi}_n(x)$  is the average of the function  $\psi(x, s)$  over the interval  $(s_n, s_{n-1})$ . Next, a certain system of elliptic equations for functions  $q_n(x)$  is derived from (6.14) using the  $s$ -dependent so-called “Carleman Weight Function”  $\exp[\lambda(s - s_{n-1})], s \in (s_n, s_{n-1})$ , where  $\lambda \gg 1, \lambda h > 1$ . Usually we use  $\lambda = 50$  in our computations.

We solve elliptic Dirichlet boundary value problems for functions  $q_n(x)$  sequentially, starting from  $q_1(x)$ . An important new element of both Section 2.9 of the book [14] and the paper [21] is the choice of the first tail function  $V_{1,1}(x)$ , see this section below. Having  $V_{1,1}(x)$ , we solve the Dirichlet boundary value problem for the function  $q_{1,1}(x)$ . Next, using (6.11) with  $V := V_{1,1}$  and substituting it in (6.10), we find the first approximation  $c_{1,1}(x)$  for our target coefficient  $c(x)$ . This is the inner iteration, i.e. when we work only inside of the domain  $\Omega$ . To perform the outer iteration in the entire space  $\mathbb{R}^3$ , we solve the forward problem (6.2), (6.3) with  $c := c_{1,1}$ , calculate the Laplace transform (6.5) at  $s := \bar{s}$  and then obtain an update  $V_{1,2}(x)$  for the tail function, using (6.12). We repeat this procedure  $m$  times until stabilization occurs, thus getting functions  $q_{1,i}, c_{1,i}, V_{1,i}, i \in [1, m]$ . The number  $m$  is found in numerical experiments. Next, we set  $q_1 := q_{1,m}, c_1 := c_{1,m}, V_{2,1} := V_{1,m}$  and repeat the same for functions  $q_{2,i}, c_{2,i}, V_{2,i}, i \in [1, m]$ . Similarly for functions  $q_{n,i}, c_{n,i}, V_{n,i}, i \in [1, m], n \in [1, N]$ .

We now describe how do we find the first tail function  $V_{1,1}(x)$ . Again following the Tikhonov concept [5, 14, 49, 70, 145], we assume that there exists unique exact solution  $c^*(x)$  of our MCIP with the noiseless data  $g^*(x, t)$  (6.4). We assume that the function  $c^*(x)$  satisfies condition (6.1). Let  $w^*(x, s)$  be the solution of the problem (6.6)-(6.7) satisfying (6.8). Using (6.12), we define the exact tail  $V^*(x, s)$  for  $s \geq \bar{s}$  as

$$V^*(x, s) = \frac{\ln w^*(x, s)}{s^2}, \forall s \geq \bar{s}.$$

Assuming that the asymptotic behavior (6.9) holds and using (6.13), we obtain

$$V^*(x, s) = \frac{p^*(x)}{s} + O\left(\frac{1}{s^2}\right), s \rightarrow \infty, x \in \overline{\Omega}. \quad (6.17)$$

for a certain function  $p^*(x)$ . We truncate the second term of this asymptotic behavior. Thus, our approximate mathematical model consists of the following assumption.

**Assumption 6.1.** There exists a function  $p^*(x) \in C^{2+\alpha}(\overline{\Omega})$  such that the exact tail function  $V^*(x, s)$  has the form

$$V^*(x, s) := \frac{p^*(x)}{s}, \quad \forall s \geq \overline{s}. \quad (6.18)$$

In addition,

$$\frac{p^*(x)}{s} = \frac{\ln w^*(x, s)}{s^2}, \quad \forall s \geq \overline{s}. \quad (6.19)$$

Since  $q^*(x, s) = \partial_s V^*(x, s)$  for  $s \geq \overline{s}$ , we derive from (6.18) that

$$q^*(x, \overline{s}) = -\frac{p^*(x)}{\overline{s}^2}. \quad (6.20)$$

Set in (6.14)  $q := q^*, V := V^*, s = \overline{s}$  and use (6.18) and (6.20). Then we obtain the following Dirichlet boundary value problem for the function  $p^*(x)$

$$\Delta p^* = 0 \text{ in } \Omega, \quad p^* \in C^{2+\alpha}(\overline{\Omega}), \quad (6.21)$$

$$p^*|_{\partial\Omega} = -\overline{s}^2 \psi^*(x, \overline{s}), \quad (6.22)$$

where  $\psi^*(x, s)$  is the exact function  $\psi(x, s)$ , which corresponds to the function  $g^*(x, t)$ . The approximate equation (6.21) is valid only within the framework of Assumption 6.1. Although this equation is linear, formula (6.16) for the reconstruction of the target coefficient  $c^*$  is nonlinear.

Assuming that the function  $\psi(x, \overline{s}) \in C^{2+\alpha}(\partial\Omega)$ , consider the solution  $p(x)$  of the following boundary value problem

$$\Delta p = 0 \text{ in } \Omega, \quad p \in C^{2+\alpha}(\overline{\Omega}), \quad (6.23)$$

$$p|_{\partial\Omega} = -\overline{s}^2 \psi(x, \overline{s}). \quad (6.24)$$

We choose the first tail function as

$$V_{1,1}(x) := \frac{p(x)}{\overline{s}}. \quad (6.25)$$

By the Schauder theorem there exists unique solution  $p$  of the problem (6.23), (6.24), see the book of Ladyzhenskaya and Uralceva [105] for Schauder theorem. Furthermore, it follows from Schauder theorem as well as from (6.21)-(6.25) that with a number  $M = M(\Omega) > 0$  the following estimate holds

$$|\nabla V_{1,1} - \nabla V^*|_{1+\alpha} \leq M \overline{s} \|\psi(x, \overline{s}) - \psi^*(x, \overline{s})\|_{C^{2+\alpha}(\partial\Omega)}. \quad (6.26)$$

Therefore, our main approximations are (6.18) and (6.19). These approximations mean the truncation of the term  $O(s^{-2})$  in (6.17). The goal of (6.18) and (6.20) is to obtain the accuracy estimate (6.26) for the first tail. We point out that these approximations are done only on the first iteration of our method. It follows from (6.26) that we obtain an approximation located in a small neighborhood of the exact solution already on the first iteration of our method, as long as the error in the boundary data  $\psi^*(x, \overline{s})$  is small. Convergence theorem 2.9.4 of [14] and theorem 5.1 of [21] guarantee that all other solutions obtained in

the iterative process also provide good approximations, as long as the number of iterations is not too large. This means that we should develop a stopping criterion numerically. The latter was done in above cited publications about this method.

Recall that the main point of any locally convergent numerical method is to get a good first guess about the solution. And this is exactly what our approximately globally convergent method delivers. Hence, we can apply the second stage of our two-stage numerical procedure to refine the solution [4, 14, 16, 17, 18, 19, 20]. More precisely, a locally convergent Adaptive Finite Element Method (adaptivity) was applied. A good numerical performance of this two-stage numerical procedure was demonstrated in [4, 14, 16, 17, 18, 19, 20], including the most challenging case of experimental data [14, 18, 20]. The adaptivity takes the solution obtained on the globally convergent stage as the first guess for further iterations. Suppose now that iterations of the first stage are stopped prior that stopping criterion is in place. Then the adaptivity still refines the solution quite well. For example, in Tests 2 and 3 on page 278 of [14] adaptivity has started from functions  $c_{n,i}(x)$  for those  $n$  which were smaller than the number  $\overline{N}$  where the stopping criterion was achieved. Nevertheless, images of these tests were quite accurate ones. Furthermore, this two-stage numerical procedure led to accurate images from experimental data, see [18] and Chapter 5 of [14]. As to some other works on the adaptivity technique, see, e.g. Beilina [12, 13] and Li, Xie and Zou [116].

**6.2. Published non-local numerical methods for MCIPs.** In parallel with the above approximately globally convergent method, another one was developed by the group of researchers from University of Texas at Arlington in collaboration with the author, see Klivanov, Su, Pantong, Shan and Liu [96], Pantong, Su, Shan, Klivanov and Liu [128], Shan, Klivanov, Su, Pantong and Liu [140], Su, Shan, Liu and Klivanov [143] and Su, Klivanov, Liu, Lin, Pantong and Liu [144]. A globally accelerated numerical method for optical tomography with continuous wave source In this case the 2d MCIP for the elliptic equation

$$\Delta u - a(x)u = -\delta(x - x_0), x \in \mathbb{R}^2, \lim_{|x| \rightarrow \infty} u(x, x_0) = 0 \quad (6.27)$$

was considered. Let  $\Gamma \subset (\mathbb{R}^2 \setminus \Omega)$  be a straight line. The MCIP consists in the determination of the unknown coefficient  $a(x)$  for  $x \in \Omega$  in (6.27) from the function  $\varphi(x, x_0) = u(x, x_0)|_{x \in \partial\Omega, x_0 \in \Gamma}$ . This is the problem of the so-called Optical Diffusion Tomography with a direct application in optical imaging of strokes in brains. In this case  $x_0$  is the position of the light source,  $u(x, x_0)$  is the light intensity and the function  $a(x)$  is proportional to the absorption coefficient of light. The most important difference between this problem and the one discussed in Section 6.1 is that the asymptotic behavior of tails like the one in (6.13) is not the case here. Thus, tails are treated quite differently in [96, 128, 140, 143, 144]. In [144] accurate images from experimental data for a phantom medium were obtained.

In works of Bikowski, Knudsen and Mueller [30], DeAngelo and Mueller [46], Hamilton, Herrera, Mueller and Von Herrmann [56] and Siltanen, Mueller and Isaacson [141] non-local reconstruction techniques are considered for the 2D problem of Electrical Impedance Tomography. In particular, images from experimental data were obtained in [46]. In papers of Alexeenko, Burov and Rumyantseva [3], Burov, Morozov and Rumyantseva [38] and Burov, Alekseenko and Rumyantsevat [39] the non-local reconstruction method of Novikov [124, 125, 126, 127] for the MCIP for an elliptic PDE with the data given in the form of the scattering amplitude is implemented. A non-local numerical method for a 2d MCIP for a hyperbolic PDE was developed by Kabanikhin and Shishlenin in [68, 69]. It is based on a 2d analog of the Gel'fand-Levitan-Krein equation. Just as the technique of [14], algorithms of [3, 38, 68, 69] also use some reasonable approximate mathematical models, which are not rigorously justified.

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